Normal trajectories in stationary spacetimes with critical asymptotic behavior

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Abstract

The aim of this note is to study the existence of normal trajectories joining two given submanifolds under the action of an external field in a stationary spacetime. Here, it is assumed that both the growth of the potential and that one of the coefficients of the metric are *critical* in a suitable sense.

1 Introduction

Given a Lorentzian manifold $(M, \langle\cdot,\cdot\rangle_L)$, a function $V \in C^1(M \times [0,T^*], \mathbb{R})$ $(T^* > 0)$, two submanifolds S, Σ of M and an *arrival time* $T \in \,]0, T^*]$, the aim of this note is to use variational tools in order to find accurate conditions ensuring that S and Σ can be connected by means of normal trajectories under the action of potential V in time T, i.e., to look for solutions of equation

$$D_s^L \dot z + \nabla_L V(z,s) = 0 \quad \text{for all } \ s \in [0,T] \tag{1.1}$$

which satisfy boundary conditions

$$\begin{cases} z(0) \in S, \ z(T) \in \Sigma, \\ \dot z(0) \in T_{z(0)}S^\perp, \dot z(T) \in T_{z(T)}\Sigma^\perp \end{cases} \tag{1.2}$$

(here, D_s^L denotes the covariant derivative along z induced by the Levi–Civita connection of metric $\langle\cdot,\cdot\rangle_L$, while $\nabla_L V(z,s)$ is the gradient of V with respect to z).

As such a problem cannot be solved in general, we limit our interest to a suitable class of Lorentzian manifolds which is "good" from a variational point of view.

Definition 1.1 *A Lorentzian manifold* $(M, \langle\cdot,\cdot\rangle_L)$*, given by a global splitting* $M = M_0 \times \mathbb{R}$*, is a* (standard) stationary *spacetime if* $(M_0, \langle\cdot,\cdot\rangle)$ *is a finite dimensional connected Riemannian manifold and its metric is*

$$\langle\zeta,\zeta'\rangle_L = \langle\xi,\xi'\rangle + \langle\delta(x),\xi\rangle\tau' + \langle\delta(x),\xi'\rangle\tau - \beta(x)\tau\tau' \tag{1.3}$$

for any $z = (x,t) \in M$ *and* $\zeta = (\xi,\tau)$*,* $\zeta' = (\xi',\tau') \in T_zM \equiv T_xM_0 \times \mathbb{R}$*, where* δ *and* β *are respectively a smooth vector field and a smooth strictly positive scalar field on* M_0*.*
In particular, M *is named* (standard) static *if* $\delta \equiv 0$*.*

Notice that the study of orthogonal trajectories joining S to Σ under the action of potential V in a stationary spacetime is interesting not only from a physical point of view, since these spacetimes represent time–independent gravitational fields (as, for example, Kerr spacetime) but also from a mathematical one.

In fact, from a variational viewpoint our problem (1.1), (1.2) is equivalent to find critical points of action functional

$$f_V(z) = \frac{1}{2}\int_0^T \langle\dot z,\dot z\rangle_L ds - \int_0^T V(z,s)\ ds$$

on a suitable set of curves (for more details, see Section 2).

Clearly, when $V \equiv 0$ and $S = \{z_0\}$, $\Sigma = \{z_1\}$ (with $z_0 = (x_0,t_0)$, $z_1 = (x_1,t_1) \in M$), the given problem reduces to the study of geodesic connectedness between z_0 and z_1 in M and, as geodesics are invariant by affine reparametrizations, arrival time T between the fixed events is not relevant (in fact, in most of the related papers it is assumed $T = 1$). In this case, in pioneer paper [9] the authors provided a variational principle in order to overcome the unboundedness of action functional f_0, so that looking for geodesics is reduced to studying critical points of the new functional

$$J(x) = \int_0^T \langle\dot x,\dot x\rangle\ ds\ +\ \int_0^T \frac{\langle\delta(x),\dot x\rangle^2}{\beta(x)}\ ds\ -\ K_t^2(x)\ \int_0^T \frac{1}{\beta(x)}\ ds \tag{1.4}$$

in $\Omega^T(x_0,x_1)$, suitable set of curves joining x_0 to x_1 in a time T, where it is

$$K_t(x) = \left(\Delta_t - \int_0^T \frac{\langle\delta(x),\dot x\rangle}{\beta(x)}\ ds\right)\ \left(\int_0^T \frac{1}{\beta(x)}\ ds\right)^{-1}, \tag{1.5}$$

with $\Delta_t = t_1 - t_0$.

According to some recent results in [2], careful estimates of J allow one to prove that z_0 and z_1 are geodesically connected in assumptions

(H_1) Riemannian manifold $(M_0, \langle\cdot,\cdot\rangle)$ is complete and smooth (at least C^3),

(H_2) there exist $\mu_1, \mu_2 \geq 0$, $k_1, k_2 \in \mathbb{R}$ and a point $\bar{x} \in M_0$ such that for all $x \in M_0$ it is

$$0 < \beta(x) \leq \mu_1 d^2(x,\bar{x}) + k_1, \tag{1.6}$$
$$\sqrt{\langle\delta(x),\delta(x)\rangle} \leq \mu_2 d(x,\bar{x}) + k_2,$$

(here, $d(\cdot,\cdot)$ denotes the distance canonically associated to Riemannian metric $\langle\cdot,\cdot\rangle$ on M_0).

More in general, if $V \equiv 0$ but $S = P \times \{t_0\}$ and $\Sigma = Q \times \{t_1\}$, where

(H_3) P and Q are two closed submanifolds of M_0 such that at least one of them is compact,

the existence of normal geodesics joining S to Σ has been studied both in a stationary spacetime, if β is far away from zero and both δ and β have a sublinear growth (see [8]), and in a static one but in growth condition (1.6) (see [6]).

On the other hand, if $V \not\equiv 0$ and $P = \{x_0\}$, $Q = \{x_1\}$, equation (1.1) has been studied not only in a Riemannian manifold (see [7]) but also both in a static and in a stationary one when potential V is time–independent, i.e.,

$$V(z,s) \equiv V(x,s) \qquad \text{for all } z = (x,t) \in M,\ s \in [0,T^*] \tag{1.7}$$

(see [1], respectively [3]). In all these cases if V satisfies assumption

(H_4) there exist $\lambda \geq 0$, $k \in \mathbb{R}$, $\bar{x} \in M_0$ such that it is

$$V(x,s) \leq \lambda d^2(x,\bar{x}) + k \qquad \text{for all } z = (x,t) \in M,\ s \in [0,T^*],$$

the existence of trajectories, which are solutions of (1.1), is not guaranteed (in fact, some counterexamples can be found, see Remark 1.4). Anyway, such solutions exist surely if coefficient λ in (H_4) and arrival time T are related by the further condition

$$\lambda T^2 < \frac{\pi^2}{2}. \tag{1.8}$$

In the more general setting, i.e., if P and Q are not a singleton but (H_3) holds, some results on a Riemannian manifold M_0 have been obtained in [5] up to assume a little bit stronger condition than (H_4):

(H_4^*) there exist $\lambda \geq 0$, $k \in \mathbb{R}$ such that

$$V(x,s) \leq \lambda d^2(x,A) + k \qquad \text{for all } z = (x,t) \in M,\ s \in [0,T^*],$$

with $A = P$ if Q is compact, or $A = Q$ if P is compact, where $d(x,A) = \inf_{z\in A} d(x,z)$.

In this note we consider problem (1.1), (1.2) on a stationary spacetime when potential V is non–trivial and time–independent.

As previously remarked, a variational formulation entirely based on the Riemannian part of the spacetime can be stated; more precisely, the given problem reduces to find critical points of functional

$$J_V(x) = \frac{1}{2}J(x) - \int_0^T V(x,s)\, ds \tag{1.9}$$

on $\Omega^T(P,Q)$, suitable set of curves joining P to Q in a time T (for the exact definition, see Section 2),

Thus, the main theorem of this note can be stated as follows.

Theorem 1.2 *Let $M = M_0 \times \mathbb{R}$ be a (standard) stationary spacetime which satisfies hypotheses (H_1), (H_2) and let $V \in C^1(M \times [0,T^*], \mathbb{R})$, $T^* > 0$, be a potential satisfying condition (1.7).*
Moreover, let $S = P \times \{t_0\}$ and $\Sigma = Q \times \{t_1\}$ be two submanifolds of M with t_0, $t_1 \in \mathbb{R}$ and P, Q two submanifolds of M_0 such to satisfy (H_3).
So, if potential V and arrival time $T \in\,]0,T^*]$ are such that (H_4^*) and (1.8) hold, S and Σ can be joined by at least one normal trajectory which solves (1.1) and (1.2).

Remark 1.3 If, in addition to the assumptions of Theorem 1.2, P and Q are both contractible in M_0, then a direct application of Ljusternik–Schnirelman Theory implies some multiplicity results either if M_0 is non–contractible in itself or if it is not (see, e.g., [6]).

Remark 1.4 Even if both P and Q are singleton, counterexamples can be construct both if hypothesis (H_4) with (1.8) fails (see, e.g., [7, Example 3.6]) and if (H_2) fails (see [4, Section 7] if β grows more than quadratically or [2, Example 2.7] if δ grows more than linearly).

2 Variational setting and abstract tools

Let $(M, \langle \cdot,\cdot \rangle_L)$ be a stationary spacetime with $M = M_0 \times \mathbb{R}$ and $\langle \cdot,\cdot \rangle_L$ as in (1.3), where $(M_0, \langle \cdot,\cdot \rangle)$ is a Riemannian manifold such that (H_1) holds. Moreover, let P and Q be two submanifolds of M_0 and fix t_0, $t_1 \in \mathbb{R}$, so that it is $S = P \times \{t_0\}$ and $\Sigma = Q \times \{t_1\}$. Now, fixed $T > 0$, for simplicity assume $I = [0,T]$.

As we want to work by means of variational tools, let us recall some basic definitions (for more details, see, e.g., [10]).

By Nash Embedding Theorem we can assume that M_0 is a submanifold of an Euclidean space $\mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ is the restriction to M_0 of the Euclidean metric of $\mathbb{R}^N$ while $d(\cdot,\cdot)$ is the corresponding distance, i.e.,

$$d(x_1,x_2) = \inf\left\{\int_a^b \sqrt{\langle\dot\gamma,\dot\gamma\rangle}ds:\ \gamma\in A_{x_1,x_2}\right\} \qquad \text{if } x_1,\ x_2\in M_0,$$

where $\gamma\in A_{x_1,x_2}$ if $\gamma:[a,b]\to M_0$ is a piecewise smooth curve joining x_1 to x_2.

Hence, it can be proved that manifold $H^1(I,M_0)$ can be identified with the set of the absolutely continuous curves $x:I\to\mathbb{R}^N$ with square summable derivative such that $x(I)\subset M_0$. Furthermore, since M_0 is a complete Riemannian manifold with respect to $\langle\cdot,\cdot\rangle$, $H^1(I,M_0)$ equipped with its standard Riemannian structure is a complete Riemannian manifold, too.

Let Z be the smooth manifold of all the $H^1(I,M)$–curves joining S to Σ while $\Omega^T(P,Q)$ denotes the smooth submanifold of $H^1(I,M_0)$ which contains all the curves joining P to Q in a time T with

$$T_x\Omega^T(P,Q) = \{\xi\in T_xH^1(I,M_0):\xi(0)\in T_{x(0)}P,\ \xi(T)\in T_{x(T)}Q\}$$

for all $x\in\Omega^T(P,Q)$. By the product structure of M, it follows

$$Z\equiv\Omega^T(P,Q)\times W^T(t_0,t_1) \quad\text{and}\quad T_zZ\equiv T_x\Omega^T(P,Q)\times H_0^1$$

for each $z=(x,t)\in Z$, as

$$W^T(t_0,t_1)=\{t\in H^1(I,\mathbb{R}):t(0)=t_0,\ t(T)=t_1\}$$

is a closed affine submanifold of $H^1(I,\mathbb{R})$ with tangent space $T_tW^T(t_0,t_1)=H_0^1$, being $W^T(t_0,t_1)=H_0^1+j^*$ with

$$j^*:s\in I\mapsto t_0+s\,\frac{\Delta_t}{T}\ \in\mathbb{R},\ H_0^1=\{t\in H^1(I,\mathbb{R}):t(0)=t(T)=0\}.$$

Let us remark that, if P and Q are closed, then submanifold Z of $H^1(I,M)$ can be equipped with the Riemannian structure

$$\langle\zeta,\zeta\rangle_H=\int_0^T\langle D_s\xi,D_s\xi\rangle ds\ +\ \int_0^T\dot\tau^2ds$$

for any $z=(x,t)\in Z$, $\zeta=(\xi,\tau)\in T_zZ$ and submanifold $\Omega^T(P,Q)$, hence Z, is complete.

Now, let $V = V(z,s)$ be a given potential on $M \times [0,T^*]$ and consider $T < T^*$. By (1.3) it follows that action integral f_V of problem (1.1), (1.2) becomes

$$f_V(z) = \frac{1}{2}\int_0^T (\langle \dot{x},\dot{x}\rangle + 2\langle \delta(x),\dot{x}\rangle \dot{t} - \beta(x)\dot{t}^2)ds - \int_0^T V(z,s)\, ds \tag{2.1}$$

if $z = (x,t) \in Z$; thus, it can be proved that boundary condition (1.2) implies that a curve $z : I \to M$ is a normal trajectory joining S to Σ if and only if $z \in Z$ is a critical point of functional f_V in Z.

As in the problem of geodesic connectedness, a way to get over the lack of boundedness of f_V on Z can be to introduce a new functional which depends only on Riemannian variable x. Clearly, such an approach is allowed if both the metric coefficients and potential V are time–independent.

Proposition 2.1 *Assume that potential V satisfies condition (1.7) and consider $z^* = (x^*,t^*) \in Z$. The following statements are equivalent:*

(i) z^* is a critical point of action functional f_V defined in (2.1);

(ii) x^* is a critical point of functional $J_V : \Omega^T(P,Q) \to \mathbb{R}$ defined in (1.9) and $t^* = \Psi(x^*)$, with $\Psi : \Omega^T(P,Q) \to W^T(t_0,t_1)$ such that

$$\Psi(x)(s) = t_0 + \int_0^s \frac{\langle \delta(x(\sigma)),\dot{x}(\sigma)\rangle}{\beta(x(\sigma))}\, d\sigma - K_t(x) \int_0^s \frac{1}{\beta(x(\sigma))}\, d\sigma$$

and $K_t(x)$ defined as in (1.5).

Moreover, for each $x \in \Omega^T(P,Q)$ and $(\xi,\tau) \in T_x\Omega^T(P,Q) \times H_0^1$ it is

$$f_V(x,\Psi(x)) = J_V(x) \qquad \textit{and} \qquad J'_V(x)[\xi] = f'_V(x,\Psi(x))[(\xi,\tau)].$$

So, from now on, assume that potential V satisfies hypothesis (1.7). Hence, by Proposition 2.1 our problem is reduced to study Riemannian functional J_V on $\Omega^T(x_0,x_1)$ and, in order to find at least one of its critical points, the following classical abstract minimum theorem is useful.

Theorem 2.2 *Let Ω be a complete Riemannian manifold and F a C^1 functional on Ω which satisfies the Palais–Smale condition, i.e., any $(x_k)_k \subset \Omega$ such that*

$$(F(x_k))_k \textit{ is bounded} \quad \textit{and} \quad \lim_{k\to+\infty} F'(x_k) = 0$$

converges in Ω up to subsequences. Then, if F is bounded from below, it attains its infimum.

3 Proof of Theorem 1.2

As already remarked in Section 2, functional J_V in (1.9) is C^1 on Riemannian manifold $\Omega^T(P,Q)$ which is complete if P and Q are closed submanifolds of M_0. Thus, in order to apply Theorem 2.2, we just need to prove that J_V is bounded from below and satisfies Palais–Smale condition. Or better, it is enough to prove that J_V is bounded from below and coercive in $\Omega^T(P,Q)$, i.e., $J_V(x_k) \to +\infty$ if $\|\dot{x}_k\| \to +\infty$ (here, $\|\cdot\|$ is the L^2–norm). In fact, if J_V is coercive in $\Omega^T(P,Q)$, then a sequence $(x_k)_k$ has to be bounded if $(J_V(x_k))_k$ is bounded, and Palais–Smale condition is a consequence of the following lemma.

Lemma 3.1 *If P and Q are two submanifolds of M_0 such that (H_3) holds, then each sequence $(x_k)_k$, bounded in $\Omega^T(P,Q)$ and such that $J'_V(x_k) \to 0$, converges up to subsequences.*

Proof. It is enough reasoning as in the proof of [10, Lemma 3.4.1] taking into account some comments in the proof of [6, Proposition 4.2]. ■

Taken any $\epsilon \in]0,1[$, it is easy to check that functional J_V can be written as

$$J_V(x) = \frac{\epsilon}{2} J^\epsilon(x) + (1-\epsilon) J_T^\epsilon(x),$$

where

$$\begin{aligned} J^\epsilon(x) \quad &= \quad \int_0^T \langle \dot{x},\dot{x}\rangle ds + \int_0^T \frac{\langle \delta(x),\dot{x}\rangle^2}{\beta^\epsilon(x)}\, ds \\ &\quad - \left(\Delta_t^\epsilon - \int_0^T \frac{\langle \delta(x),\dot{x}\rangle}{\beta^\epsilon(x)}\, ds\right)^2 \left(\int_0^T \frac{1}{\beta^\epsilon(x)}\, ds\right)^{-1}, \end{aligned}$$

with $\beta^\epsilon(x) = \epsilon\, \beta(x)$ and $\Delta_t^\epsilon = \frac{\Delta_t}{\epsilon}$, and

$$J_T^\epsilon(x) = \frac{1}{2}\int_0^T \langle \dot{x},\dot{x}\rangle ds - \int_0^T V^\epsilon(x,s) ds, \qquad \text{with } V^\epsilon(x,s) = \frac{V(x,s)}{1-\epsilon}.$$

Then, the following lemmas can be stated.

Lemma 3.2 *If (H_2) and (H_3) hold, then for each $\epsilon \in]0,1[$ functional J^ϵ is bounded from below and coercive in $\Omega^T(P,Q)$.*

Proof. The proof can be obtained by reasoning as in the proofs of [2, Lemma 2.6] and [6, Proposition 4.1] with some minor changes according to assume P or Q as compact set. ■

Lemma 3.3 *If* (H_3), (H_4^*) *and (1.8) hold, then, taken* $\epsilon \in]0,1[$ *small enough so that*

$$\frac{\lambda}{1-\epsilon} T^2 < \frac{\pi^2}{2},$$

functional J_T^ϵ *is bounded from below and coercive in* $\Omega^T(P,Q)$.

Proof. See [5, Lemma 3.1]. ■

Obviously, in the hypotheses of Theorem 1.2, Lemmas 3.2 and 3.3 imply that functional J_V is bounded from below and coercive in $\Omega^T(P,Q)$; hence, it satisfies Palais–Smale condition (see Lemma 3.1) and Theorem 2.2 applies. So, J_V attains its infimum, and, thus, a solution of the given problem must exist. ■

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