The lightlike dimensional reduction of the inhomogeneous massless little group into the Galilei group

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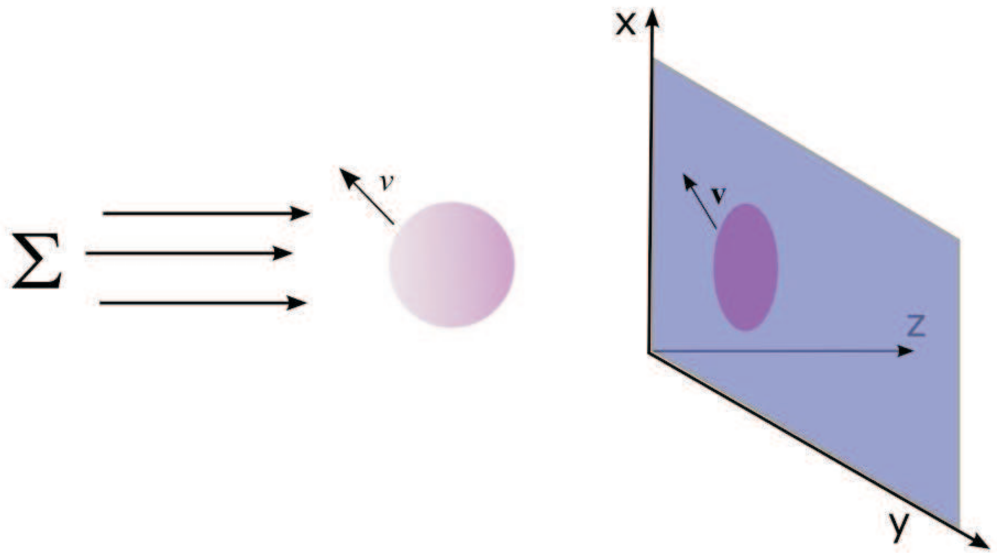
Talk based on results by the author and on previous results by Eisenarth, Lichnerowicz, Bargmann, Levy-Leblond, Duval, Burdet, Künzle, Perrin, Gibbons and Horváthy.

The Galilean group and the transformation of shadows in special relativity, gr-qc/0510063

Shadows from a distance source Σ

- Our aim is to study the shadows of a monochromatic source of light Σ that emits photons of momentum $p^\mu = \omega n^\mu$, $n^\mu = (1, 0, 0, 1)$.



- How does it changes under frame boosts that preserve the direction of p?

The Poincaré group

The Poincare group is ($\eta_{00} = -1$, $\mu = 0, \dots, d+1$)

$$\begin{aligned} [J^{\alpha\beta}, J^{\gamma\delta}] &= \eta^{\alpha\delta} J^{\beta\gamma} + \eta^{\beta\gamma} J^{\alpha\delta} - \eta^{\alpha\gamma} J^{\beta\delta} - \eta^{\beta\delta} J^{\alpha\gamma}, \\ [P^{\alpha}, J^{\beta\gamma}] &= \eta^{\alpha\beta} P^{\gamma} - \eta^{\alpha\gamma} P^{\beta}. \end{aligned}$$

defining ($a = 1, \dots, d$)

$$W_a = J^{0\,a} - J^{d+1\,a} = -n_\mu J^{\mu a},$$

we obtain that generators J_{ab}, W_a generate a $\mathcal{ISO}(d)$ subalgebra

$$\begin{aligned} [J_{ab}, J_{cd}] &= \delta_{ad} J_{bc} + \delta_{bd} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac}, \\ [W_a, J_{bc}] &= \delta_{ab} W_c - \delta_{ac} W_b. \end{aligned}$$

The Little group

The infinitesimal Lorentz transformation $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$ with

$$\Lambda^{\alpha}_{\ \beta} = (I + \frac{1}{2}\Omega_{\mu\nu} J^{\mu\nu})^{\alpha}_{\ \beta}$$

leads to the Lie algebra representation $(J^{\mu\nu})^{\alpha}_{\ \beta} = \eta^{\mu\alpha}\delta^{\nu}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}_{\beta}$, it follows

$$\begin{aligned}
(J_{ab})^{\alpha}_{\ \beta} &= \delta^{\alpha}_{a}\delta_{b\beta} - \delta^{b}_{\alpha}\delta_{a\beta} \\
(W_a)^{\alpha}_{\ \beta} &= \eta^{a\alpha} n_{\beta} - n^{\alpha}\delta^{a}_{\beta}
\end{aligned}$$

and

$$(J_{ab})^{\alpha}_{\ \beta} n^{\beta} = (W_a)^{\alpha}_{\ \beta} n^{\beta} = 0$$

thus the generators J_{ab}, W_a leave n^{μ} invariant: they generate the little group $L(e_{d+1})$. The generators W_a are often called the *translational* generators. On the contrary we shall see that they generate Galilean boosts.

The Little group II

The Little group transformation $x'^{\mu} = L^{\mu}_{\ \nu} x^{\nu}$ takes the form

$$L^{\mu}_{\ \nu} = \begin{pmatrix} 1+\zeta & -\alpha^c \mathrm{R}_{cb} & -\zeta \\ -\alpha^a & \mathrm{R}^a_{\ b} & \alpha^a \\ \zeta & -\alpha^c \mathrm{R}_{cb} & 1-\zeta \end{pmatrix},$$

with $\zeta = \alpha^a \alpha_a / 2$. The infinitesimal transformation is

$$I + \frac{1}{2} \Omega_{ab} J^{ab} + \alpha^a W_a$$

It means that $\Omega_{0a} = -\Omega_{d+1\,a} = \alpha_a$, $\Omega_{0\,d+1} = 0$, or *in order to keep the null momentum* $p = \omega n$ *invariant the frames must rotate and accelerate in such a way that the (i) acceleration along* e_{d+1} *vanishes (ii) the acceleration and the angular velocity perpendicular to* e_{d+1} *must be equal in magnitude and perpendicular to each other.*

The inhomogeneous Little group

Let us define

$$\mathcal{M} = P^{\mu} n_{\mu} = P^{d+1} - P^{0}$$

and $H = P^0$ then the generators $\mathcal{M}$, H, P^a, W_a and J_{ab} span a Lie algebra whose infinitesimal Poincaré transformations leave n^{μ} invariant and include the translations

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{ad} J_{bc} + \delta_{bd} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac}, \\
[W_a, J_{bc}] &= \delta_{ab} W_c - \delta_{ac} W_b, \\
[P_a, J_{bc}] &= \delta_{ab} P_c - \delta_{ac} P_b, \\
[W_a, H] &= P_a, \\
[W_a, P_b] &= \delta_{ab} \mathcal{M}.
\end{aligned}$$

$\mathcal{M}$ generates translations along n^{μ}. The infinitesimal transformation of $x'^{\mu} = L^{\mu}_{\ \nu} x^{\nu} - a^{\mu}$ is

$$I + \frac{1}{2}\Omega_{ab} J^{ab} + \alpha^{a} W_a - a^{b} P_b + (a^{0} - a^{d+1}) H + a^{d+1} \mathcal{M}$$

The Galilean group

The *shadow mass* generator $\mathcal{M}$ spans an ideal I and $N = \exp I$ is a normal subgroup. Moreover $\mathcal{M}$ belongs to the center of $\mathcal{IL}(\mathbf{e}_{d+1})$. The quotient group $IL(n)/N$ is isomorphic to the Galilean group $Gal(d)$ in d+1 spacetime dimensions. We have the Lie algebra central extension

$$0 \to \{\mathcal{M}\} \to \mathcal{IL}(\mathbf{e}_{d+1}) \to \mathcal{G}al(d) \to 0,$$

Thus the inhomogeneous Little group is a particular central extension of the Galilean group

$$1 \to N(\sim T_1) \to IL(\mathbf{e}_{d+1}) \to Gal(d) \to 1.$$

To the same Lie algebra central extension there corresponds another central extension

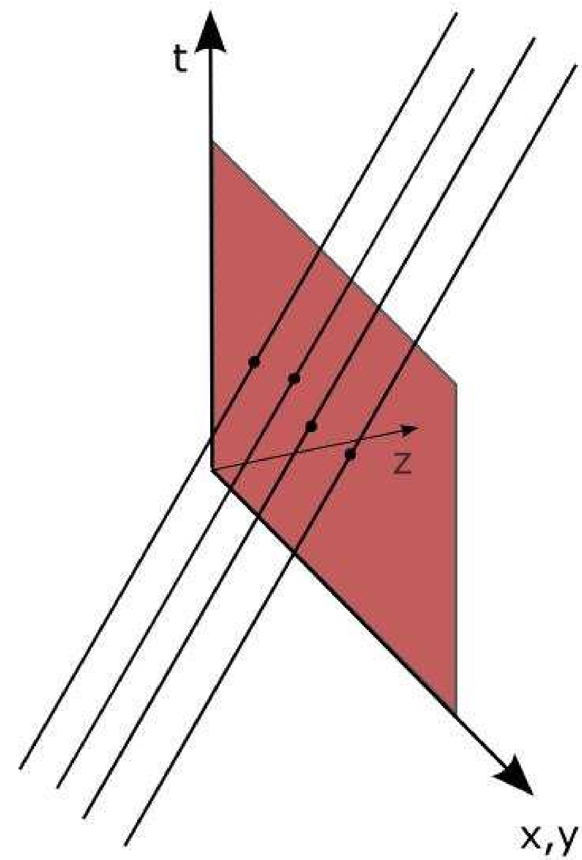
$$1 \to U(1) \to B(\mathbf{e}_{d+1}) \to Gal(d) \to 1.$$

$B(\mathbf{e}_{d+1})$ is the Bargmann group. It is useful in quantum mechanics.

The fiber bundle with null fibration $M \to Q$

The Galilean group $Gal((d) \sim IL/N$ acts unambiguously on the d+1 quotient space-time $Q = M/N$ made of 'events' Nx. Thus the events of Q are the light rays of direction n.

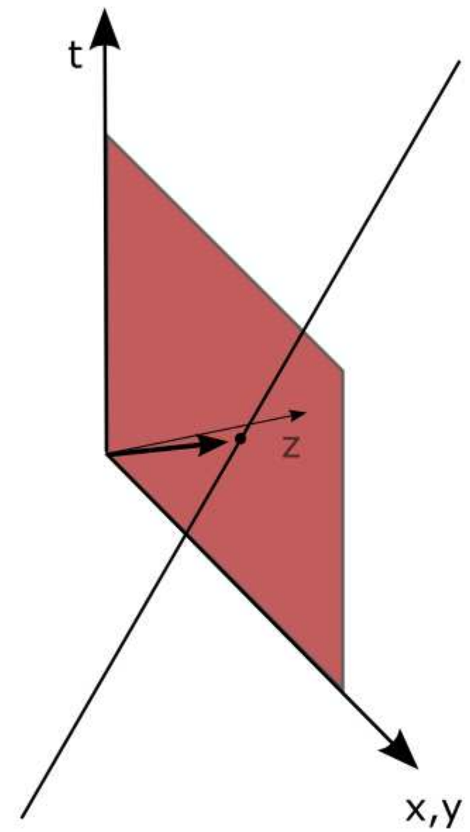
The Bargmann bundle is obtained identifying points x and x' such that $x' - x = 2\pi k$, $k \in \mathbb{N}$. The structure group is the Bargmann group.



The relation with shadows

An event x has a shadow on each screen perpendicular to the direction of light. Each shadow is a representant of the point on Q, Nx. Let t be the hitting time, x^a the hitting point ($x^0 = t + x^{d+1}$) then x^μ admits the unique decomposition

$$x^\mu = \begin{pmatrix} x^{d+1} + t \\ x^a \\ x^{d+1} \end{pmatrix} = x^{d+1} n^\mu + \begin{pmatrix} t \\ x^a \\ 0 \end{pmatrix}$$



The transformation of shadows

If $x'^{\mu} = L^{\mu}_{\ \nu} x^{\nu} - a^{\mu}$ (inhomogeneous Little group transformation) then

$$x'^{\mu} = \begin{pmatrix} x'^{d+1} + t' \\ \mathbf{x}' \\ x'^{d+1} \end{pmatrix} = x'^{d+1} n^{\mu} + \begin{pmatrix} t' \\ \mathbf{x}' \\ 0 \end{pmatrix}$$

with

$$x'^{d+1} = x^{d+1} + t\zeta - \alpha^{a} \mathrm{R}_{ab} x^{b} - a^{d+1},$$

and (Galilei transformation)

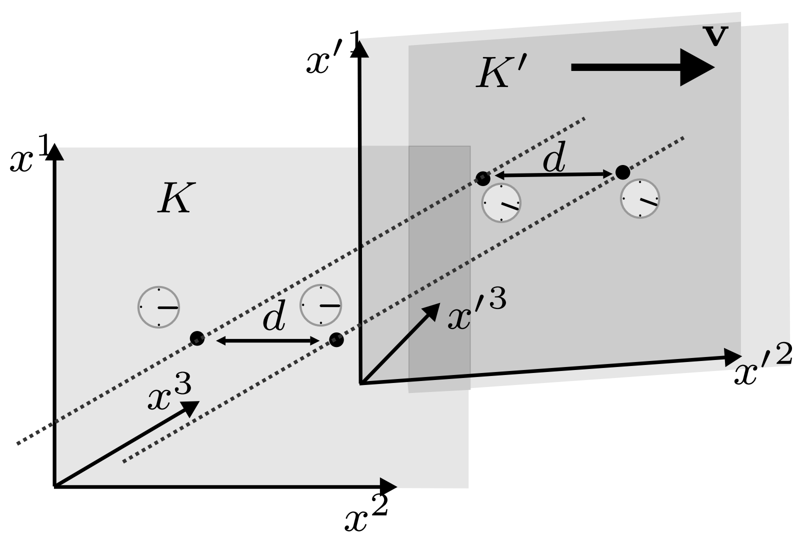
$$\begin{aligned} t' &= t - (a^{0} - a^{d+1}), \\ x'^{b} &= \mathrm{R}^{b}_{\ c} x^{c} - t\alpha^{b} - a^{b}. \end{aligned}$$

Recall

$$I + \frac{1}{2}\Omega_{ab} J^{ab} + \alpha^{a} W_{a} - a^{b} P_{b} + (a^{0} - a^{d+1}) H + a^{d+1} \mathcal{M}$$

The transformation of shadows

- The light spot of a beam of light or the dark spot of a screened beam of light transform, changing inertial frame, with a Galilean transformation.
- Since the shadows are made by dark spots their points transform with a Galilean transformation, hence the shadow have the same shape (but different velocity) in all perpendicular to light screens.
- The transformation of hitting time t implies an absolute (Galilean) simultaneity: if two light beams hit a perpendicular-to-light screen at the same time then they hit all perpendicular-to-light screens at the same time.



The shadow of a pointlike relativistic particle...

In the coordinates x^{d+1}, t and x^a the Minkowski metric reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 - 2\mathrm{d}t\,\mathrm{d}x^{d+1} + \mathrm{d}x^a \mathrm{d}x_a .$$

Hamilton's principle in configuration space

$$0 = \delta \int m \mathrm{d}\tau = \delta \int_{t_0}^{t_1} \mathcal{L} \mathrm{d}t = \delta \int_{t_0}^{t_1} m \sqrt{2\dot{x}^{d+1} + 1 - \dot{\mathbf{x}}^2}\, \mathrm{d}t.$$

In the presence of more particles the total Lagrangian $\mathcal{L}$ is the sum of Lagrangians $\mathcal{L}_{(i)}$. The cyclic variables $x_{(i)}^{d+1}$ can be removed using Routh's reduction. The conserved conjugated momenta (*shadow mass*) are

$$\mu_{(i)} = p_{(i)}^0 - p_{(i)}^{d+1}.$$

The reduced variational principle is $\delta \int_{t_0}^{t_1} R \mathrm{d}t = 0$, where the Routhian R is given by

$$R(\mathbf{x}_{(i)}, \dot{\mathbf{x}}_{(i)}) = \sum_{(i)} [\mu_{(i)} \dot{x}_{(i)}^{d+1} - \mathcal{L}_{(i)}] = \sum_{(i)} [T_{(i)} - V_{(i)}].$$

... behaves as a classical non-relativistic particle

The shadow kinetic and internal energy are

$$
\begin{aligned}
T &= \frac{\mu}{2}\dot{\mathbf{x}}^2 = \frac{p^a p_a}{2(p^0 - p^{d+1})}, \\
V &= \frac{1}{2}[\frac{m^2}{\mu} + \mu] = \frac{1}{2}[\frac{m^2}{p^0 - p^{d+1}} + p^0 - p^{d+1}] \\
E &= T + V = \frac{\mu}{2}\dot{\mathbf{x}}^2 + \frac{1}{2}[\frac{m^2}{\mu} + \mu] = p^0.
\end{aligned}
$$

One expects the shadow worldline to behave as a classical particle of (shadow) mass μ. Let us verify this fact in the shadow of a relativistic collision.

$$
\sum_{(i)}^{N} p^{\mu}_{(i)} = \sum_{(i)}^{\bar{N}} \bar{p}^{\mu}_{(i)}.
$$

Conservation principles

The particle worldlines are geodesics that project into geodesics of Q: the shadow worldlines. We obtain

$$\sum_{(i)}^{N} \mu_{(i)} = \sum_{(i)}^{\bar{N}} \bar{\mu}_{(i)},$$

that is, the total shadow mass is conserved in the shadow collision.

$$\sum_{(i)}^{N} \mu_{(i)} \dot{\mathbf{x}}_{(i)} = \sum_{(i)}^{\bar{N}} \bar{\mu}_{(i)} \dot{\bar{\mathbf{x}}}_{(i)},$$

that is, the shadow momentum is conserved. Finally

$$\sum_{(i)}^{N} E_{(i)} = \sum_{(i)}^{\bar{N}} \bar{E}_{(i)},$$

the shadow energy is conserved.

Conservation of kinetic energy

The total kinetic energy in conserved

$$\sum_{(i)}^{N} \frac{\mu_{(i)}}{2}\dot{\mathbf{x}}_{(i)}^{2} = \sum_{(i)}^{N} \frac{\mu_{(i)}}{2}\dot{\bar{\mathbf{x}}}_{(i)}^{2}.$$

if the total internal energy is conserved.

- The shadow of an elastic relativistic collision is an elastic classical collision provided the shadow masses are preserved in the collision, $\mu_{(i)} = \bar{\mu}_{(i)}$, $(i) = 1, \dots, N$.
- The shadow of a collision that involves only massless particles is a classical elastic collision.

The inverse problem

Assume the conservation of total mass, momentum and kinetic energy on Q

$$\sum_{(i)}^{N} \mu_{(i)} = \sum_{(i)}^{\bar{N}} \bar{\mu}_{(i)}; \qquad \sum_{(i)}^{N} \mu_{(i)} \dot{\mathbf{x}}_{(i)} = \sum_{(i)}^{\bar{N}} \bar{\mu}_{(i)} \dot{\bar{\mathbf{x}}}_{(i)}; \qquad \sum_{(i)}^{N} \frac{1}{2} \mu_{(i)} \dot{\mathbf{x}}^2_{(i)} = \sum_{(i)}^{\bar{N}} \frac{1}{2} \bar{\mu}_{(i)} \dot{\bar{\mathbf{x}}}^2_{(i)}$$

we want to construct a Galilei invariant *lift* of the collision to M. Set $m = \alpha \mu$.

Lightlike 0-lift. *The elastic collision on Q can be regarded as the shadow of a collision on M in which only massless particles of momentum*

$$p^{\mu}_{(i)} = \frac{\mu_{(i)}}{2} \begin{pmatrix} \dot{\mathbf{x}}^2 + 1 \\ 2\dot{\mathbf{x}} \\ \dot{\mathbf{x}}^2 - 1 \end{pmatrix}_{(i)},$$

are involved.

Timelike 1-lift. *The elastic collision on Q can be regarded as the shadow of a collision between particles of momentum*

$$p^{\mu}_{(i)} = \frac{\mu_{(i)}}{2} \begin{pmatrix} \dot{\mathbf{x}}^2 + 2 \\ 2\dot{\mathbf{x}} \\ \dot{\mathbf{x}}^2 \end{pmatrix}_{(i)},$$

for which the shadow mass coincide with mass.

Conclusions

- The relation of the inhomogeneous Little group and the Galilean group has been studied in detail.
- An application has been found in the transformation of shadows between screens perpendicular to the direction of light and at rest in different inertial frames.
- The shadow of a relativistic particle has been proved to behave as a classical particle with a suitable shadow mass, as could have been expected from the fact that the Galilean invariance in the reduced space is inherited from the original Poincaré invariance.
- The projection of a relativistic collision and the inverse process of lifting a shadow have been investigated.