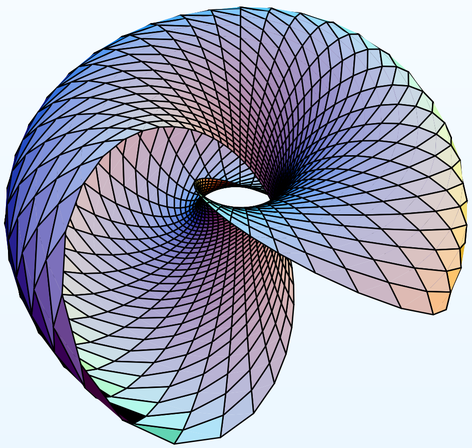

Solitons in the 2-dimensional $O(3)$ Nonlinear Sigma Model & $O_1(3)$ Nonlinear Sigma Model foliated by circles

Magdalena Caballero.

Joint work with Manuel Barros and Miguel Ortega

Departamento de Geometría y Topología.

Universidad de Granada

From now on we will write NSM_2 *instead of* 2*-dimensional Nonlinear Sigma Model.*

Index

Index

- Introduction
  - The problem in $\mathbb{R}^3$.
  - The problem in $\mathbb{L}^3$.
  - Why do we study these problems?

Index

- Introduction
  - The problem in $\mathbb{R}^3$.
  - The problem in $\mathbb{L}^3$.
  - Why do we study these problems?
- The $O(3)$ NSM_2
  - The problem.
  - Its conformal invariance.
  - Solitons with rotational symmetry.
  - Villarceau circles (VC).
  - Solitons folliated by VC.

Index

- Introduction
  - The problem in $\mathbb{R}^3$.
  - The problem in $\mathbb{L}^3$.
  - Why do we study these problems?
- The $O(3)\ NSM_2$
  - The problem.
  - Its conformal invariance.
  - Solitons with rotational symmetry.
  - Villarceau circles (VC).
  - Solitons folliated by VC.
- The $O_1(3)\ NSM_2$ and the $O_1(3)\ NSM_{(1+1)}$
  - The problem.
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Introduction

Introduction: The problem in $\mathbb{R}^3$

Let M be a differential surface (compact). We consider the class of ***immersions of*** M ***in*** $\mathbb{R}^3$ with ***the same boundary*** and ***the same Gauss map along the common boundary***.

Needless to say, we are using the standard metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^3$.

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Introduction: The problem in $\mathbb{R}^3$

The critical points of $\mathcal{S}$ are called solitons in the $O(3)$ NSM_2.
Let identify Φ with $\Phi(M)$.

1. **Are there any solitons in the $O(3)$ $\mathbf{NSM_2}$ foliated by parallel circles?**
2. **Are there any solitons in the $O(3)$ $\mathbf{NSM_2}$ foliated by nonparallel circles?**

Parallel circles$=$circles contained in parallel planes.

Nonparallel circles$=$circles contained in nonparallel planes.

Introduction: The problem in $\mathbb{L}^3$

Let M be a differential surface (compact). We consider the class of ***immersions of M in $\mathbb{L}^3$*** with ***the same boundary*** and ***the same Gauss map along the common boundary***

We are going to study immersions that give a revolution surface in $\mathbb{L}^3$ around the x_3-axis, so the common boundary must be space-like.

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We have considered two spaces of ***immersions of M in $\mathbb{L}^3$*** with ***the same space-like boundary*** and ***the same Gauss map along the common boundary***.

- The first one consist of immersions whose Gauss map is time-like, and so M with the corresponding induced metric is Riemannian.

  The study of the functional $\mathcal{S}$ in this space is called the $O_1(3)$ NSM_2. The critical points of $\mathcal{S}$ are solitons.

Introduction: The problem in $\mathbb{L}^3$

- The second one consist of immersions whose Gauss map is space-like, and so M with the corresponding induced metric is Lorentzian.
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The existence of solitons of these models admitting other foliations is something we are now studying. I hope next year I will be here speaking about them.

Introduction: Why do we study these problems?

Why do we study these Nonlinear Sigma Models?

Physical motivations: these Nonlinear Sigma models are ubiquitous in physics, from Condensed Matter Physics to High Energy Physics. They plays an important role in string theories.

They have their own interest in Differential Geometry: The $O(3)$ NSM_2 belongs to a family of variational problems associated with functionals of the type

$$\mathcal{B}(M) = \int_M F(dN) dA,$$

acting on a certain class, $\mathcal{C}$, of surfaces, M, in $\mathbb{R}^3$ with Gauss map N, where F is a given function and dA is the element of area, on M, of the induced metric from the Euclidean one. As the Willmore problem and the Plateau problem.

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Why are we interested in solitons foliated by circles?

We know the Plateau problem with boundary has solutions foliated by parallel circles but it hasn´t solutions foliated by nonparallel circles. What about our problem?

The $O(3)$ NSM_2

The $O(3)\ NSM_2$

Let M be a compact, differential surface with $\partial M = c_1 \cup c_2 \cdots \cup c_n$.
The first order boundary conditions are (Γ, N_o), where

1. $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ is a finite set of disjoint regular closed curves in $\mathbb{R}^3$.
2. N_o is a unit normal vector field along Γ such that $\langle N_o(x), \Gamma'(x)\rangle = 0$.

We define $I_\Gamma(M, \mathbb{R}^3) := \{\phi : M \to \mathbb{R}^3$ inmersion $/\phi(c_j) = \gamma_j, 1 \leq j \leq n$ and $d\phi_p(T_p M) \perp N_o(\phi(p)), \forall p \in \partial M\}$.

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The field configuration of the $\mathbf{NSM}_2$ can be identified with $I_\Gamma(M, \mathbb{R}^3)$ and the Lagrangian that governs the dynamics of the model is $\mathcal{S} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}$

$$\boxed{\mathcal{S}(\phi) = \int_M \mid dN_\phi \mid^2 \ dA_\phi,}$$

where dA_ϕ is the element of area of $(M, \phi^*\langle\cdot,\cdot\rangle)$, $N_\phi : M \to \mathbb{S}^2$ is the Gauss map of ϕ.

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Let denote by H_ϕ the mean curvature function of $\phi \in I_\Gamma(M, \mathbb{R}^3)$ and by $G_\phi = \det(-dN_\phi)$ the Gaussian curvature of $(M, \phi^* g)$.

$O(3)\ NSM_2$: *Its conformal invariance*

1. The case where $\partial M = \emptyset$ and M is compact can be regarded as a particular one

$$\left.\begin{array}{r} \mid dN_\phi \mid^2 = 4H_\phi^2 - 2G_\phi \\ \text{Gauss-Bonnet} \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \phi \text{ critical point of } \mathcal{S} : I(M, \mathbb{R}^3) \to \mathbb{R} \Leftrightarrow \\ \phi \text{ critical point of } \mathcal{W} : I(M, \mathbb{R}^3) \to \mathbb{R}, \\ \mathcal{W}(\phi) = \int_M H_\phi^2 dA_\phi \end{array}}$$

2. When M is compact and not boundary free, and for any (Γ, N_o)

$$\left.\begin{array}{r} \mid dN_\phi \mid^2 = 4H_\phi^2 - 2G_\phi \\ \text{Boundary conditions} \\ \text{Gauss-Bonnet} \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \phi \text{ critical point of } \mathcal{S} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R} \Leftrightarrow \\ \phi \text{ critical point of } \mathcal{W} : I_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}, \\ \mathcal{W}(\phi) = \int_M H_\phi^2 dA_\phi + \int_{\phi(\partial M)} \kappa^\phi ds \end{array}}$$

$O(3)\ NSM_2$: *Solitons with rotational symmetry*

Are there any solitons in the $O(3)\ NSM_2$ foliated by parallel circles?

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Are there any solitons in the $O(3)\ NSM_2$ foliated by parallel circles? The answer is YES. The following theorem gives us the space of solitons with rotational symmetry so, in particular, they are foliated by parallel circles.

Theorem 1 [M. Barros] *The solitons in the* $O(3)\ \mathbf{NSM}_2$ *(with boundary) with rotational symmetry correspond with the surfaces of revolution obtained by rotation of clamped elastic curves in the hyperbolic plane.*

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Clamped elastic curves in the hyperbolic plane P:

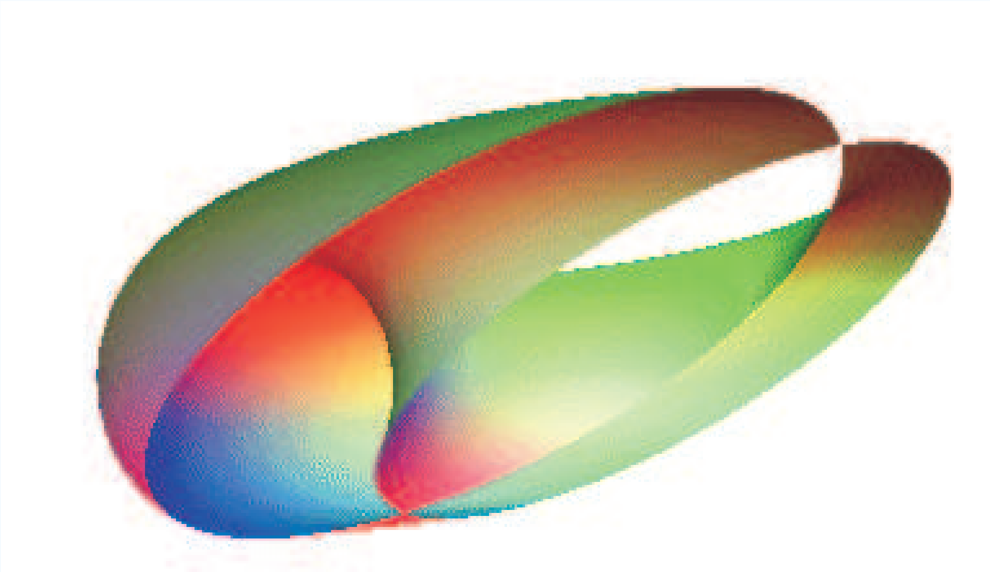
We choose two points in P, m_1 and m_2, unit vectors $\nu_i \in T_{m_i}P$, $i = 1, 2$, and the space or clamped curves $\Lambda = \{\alpha : [t_1, t_2] \longrightarrow P,\ \alpha(t_i) = m_i,\ \alpha'(t_i) = \nu_i,\ i = 1, 2\}$.

In this setting we take the variational problem associated with the total elastic energy, $\mathcal{E} : \Lambda \to \mathbb{R}$, given by

$$\mathcal{E}(\alpha) = \int_\alpha \kappa^2\, ds.$$

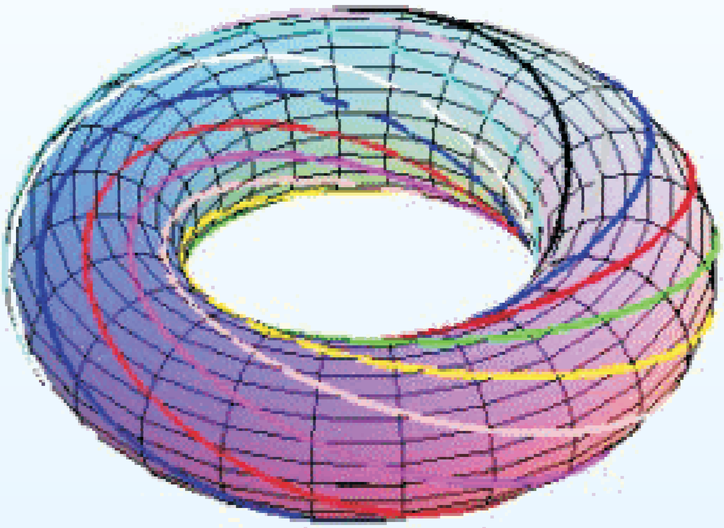
$O(3)\ NSM_2$: *Villarceau circles (VC)*

First kind Villarceau circles (1VC) and Second kind Villarceau circles (2VC) can be found by intersecting a torus $\mathbf{T}$ with a bitangent plane.



$\mathbf{1}$**VC** and $\mathbf{2}$**VC** are two families of nonparallel circles.

$O(3)\ NSM_2$: *Villarceau circles (VC)*



Given a torus, two Villarceau circles of the same kind are always linked.

$O(3)\ NSM_2$: Villarceau circles (VC)

Let $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 \ : \ |z|^2 = 1\}$ be the unit tree sphere. We consider the usual Hopf map, $\Pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$, $\Pi(z_1, z_2) = (z_1 \bar{z_2}, \frac{1}{2}(|z_1|^2 - |z_2|^2))$. And the following isometry $F : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$, $F(z_1, z_2) = (z_1, \bar{z_2})$.

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- FIRST KIND VILLARCEAU CIRCLES: The fibres of Π can be projected down to $\mathbb{R}^3$ by the stereographic projection, E_o, obtaining that they are just the **1VC** over a family of revolution tori around the $\{x_3\}$ - axis.

  The **1VC** are also the orbits in $\mathbb{R}^3 - (\{x_3\} - axis)$ associated with the group of preserving orientation conformal mappings in $\mathbb{R}^3 - (\{x_3\} - axis)$

  $$\mathbf{H} = \{\psi_t = E_o \circ \varphi_t \circ E_o^{-1} \ : \ t \in \mathbb{R}\},$$

  where $\varphi_t(z) := e^{it} \cdot z$ for all $z \in \mathbb{S}^3$.

$O(3)\ NSM_2$: *Villarceau circles (VC)*

Let $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 \ : \ |z|^2 = 1\}$ be the unit tree sphere. We consider the usual Hopf map, $\Pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$, $\Pi(z_1, z_2) = (z_1 \bar{z_2}, \frac{1}{2}(|z_1|^2 - |z_2|^2))$. And the following isometry $F : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$, $F(z_1, z_2) = (z_1, \bar{z_2})$.

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  where $\varphi_t(z) := e^{it} \cdot z$ for all $z \in \mathbb{S}^3$.

- SECOND KIND VILLARCEAU CIRCLES: The fibres of $\Pi \circ F$ can be projected down to $\mathbb{R}^3$ by the stereographic projection, E_o, obtaining that they are just the **2VC** over a family of revolution tori around the $\{x_3\}$ - axis.

  Another action for **2VC** can be defined. However, we can reduce all computations to the case of **1VC** by using the isometry F.

$O(3)$ NSM_2: *Solitons folliated by VC*

Are there any solitons in the $O(3)$ NSM_2 foliated by nonparallel circles?

$O(3)\ NSM_2$: *Solitons folliated by VC*

Are there any solitons in the $O(3)\ NSM_2$ foliated by nonparallel circles? We answer this question in the affirmative by showing the moduli space of solitons foliated by VC.

Theorem 2 [M. Barros, _, M. Ortega] *The solitons in the $O(3)$ $\mathbf{NSM_2}$ (with boundary) that are foliated by First Kind VC, correspond with the surfaces obtained as stereographic projection of Hopf tubes on clamped elastic curves in the once punctured two sphere with ratio* $1/2$.

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Theorem 3 [M. Barros, _, M. Ortega] *The solitons in the $O(3)$ $\mathbf{NSM}_2$ (with boundary) that are foliated by Second Kind VC, correspond with the surfaces obtained as stereographic projection of liftings of clamped elastic curves in the once punctured two sphere with ratio $1/2$ via the map $\Pi_- : \mathbb{S}^3 \to \mathbb{S}^2(1/2), \quad \Pi_-(z_1, z_2) = (z_1 z_2, \frac{1}{2}(|z_1|^2 - |z_2|^2))$.*

$O(3)\ NSM_2$: *Solitons foliated by VC*

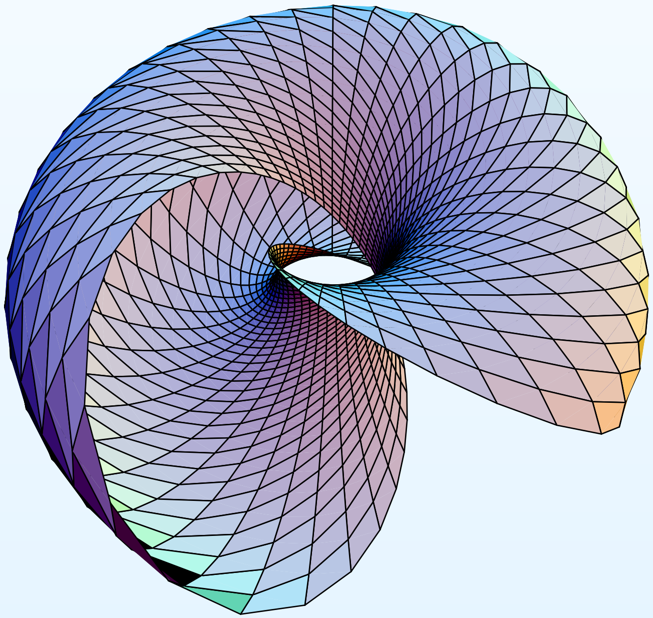
Clamped elastic curves in the once punctured two sphere $\mathbb{S}^{\mathbf{2}}(\frac{\mathbf{1}}{\mathbf{2}}) - \{\mathbf{m}\}$:

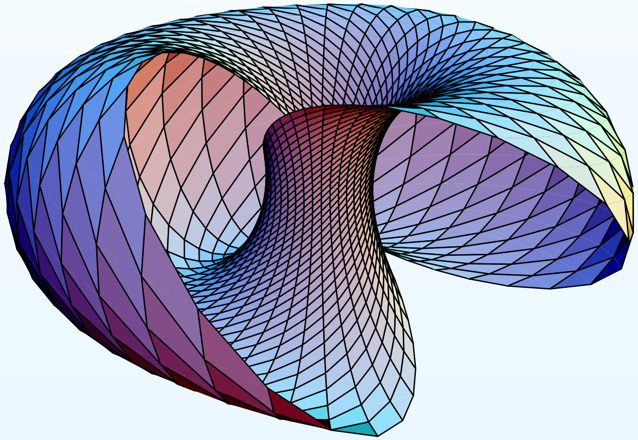
We choose two points in $\mathbb{S}^2(\frac{1}{2}) - \{m\}$, m_1 and m_2, unit vectors $\nu_i \in T_{m_i}(\mathbb{S}^2(\frac{1}{2}) - \{m\})$, $i = 1, 2$, and the space or clamped curves $\Lambda = \{\alpha : [t_1, t_2] \longrightarrow \mathbb{S}^2(\frac{1}{2}) - \{m\},\ \alpha(t_i) = m_i,\ \alpha'(t_i) = \nu_i,\ i = 1, 2\}.$
In this setting we take the variational problem associated with the total elastic energy, $\mathcal{E} : \Lambda \to \mathbb{R}$, given by

$$\mathcal{E}(\alpha) = \int_\alpha (\kappa^2 + 4)\, ds.$$

$O(3)\ NSM_2$: *Solitons foliated by VC*

Let see a soliton foliated by Villarceau circles:





It seems to be part of a revolution torus, but it is not.

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$

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Let M be a compact, differential surface with $\partial M = c_1 \cup c_2 \cdots \cup c_n$.
The first order boundary conditions are (Γ, N_o), where

1. $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$ is a finite set of space-like disjoint regular closed curves in $\mathbb{L}^3$.
2. N_o is a unit normal vector field along Γ such that $\langle N_o(x), \Gamma'(x)\rangle = 0$ and $\langle N_o(x), N_o(x)\rangle = \epsilon$.

Where $\langle \cdot, \cdot \rangle$ is the standard metric in $\mathbb{L}^3$.

For $O_1(3)\ NSM_2 \longrightarrow \boxed{\epsilon = -1}$

For $O_1(3)\ NSM_{(1+1)} \longrightarrow \boxed{\epsilon = 1}$

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For $O_1(3)\ NSM_2 \longrightarrow \boxed{\epsilon = -1}$

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We define $I_\Gamma^\epsilon(M, \mathbb{R}^3) := \{\phi : M \to \mathbb{R}^3$ inmersion $/\phi(c_j) = \gamma_j, 1 \le j \le n$, $d\phi_p(T_pM) \perp N_o(\phi(p)), \forall p \in \partial M$ and $\phi^*\langle\cdot,\cdot\rangle$ is riemanian if $\epsilon = -1$ and lorentzian if $\epsilon = 1\}$.

The field configuration of the $O_1(3)\ NSM_2$ can be identified with $I_\Gamma^{-1}(M, \mathbb{R}^3)$.
The field configuration of the $O_1(3)\ NSM_{(1+1)}$ can be identified with $I_\Gamma^{1}(M, \mathbb{R}^3)$.

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$

The Lagrangian that governs the dynamics of both models is $\mathcal{S} : I^{\epsilon}_{\Gamma}(M, \mathbb{R}^3) \rightarrow \mathbb{R}$

$$\boxed{\mathcal{S}(\phi) = \int_M \mid dN_\phi \mid^2 \ dA_\phi,}$$

where dA_ϕ is the element of area of $(M, \phi^* \langle \cdot, \cdot \rangle)$, $-dN_\phi$ is the shape operator.

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Let denote by H_ϕ the mean curvature function of $\phi \in I_\Gamma(M, \mathbb{R}^3)$ and by $G_\phi = \epsilon \operatorname{det}(-dN_\phi)$ the Gaussian curvature of $(M, \phi^* g)$.
There exists a relation between them, similar as the classical one for surfaces immersed in $\mathbb{R}^3$:

$$\mid dN_\phi \mid^2 = 4H_\phi^2 - 2\epsilon G_\phi$$

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where dA_ϕ is the element of area of $(M, \phi^*\langle\cdot,\cdot\rangle)$, $-dN_\phi$ is the shape operator.

Let denote by H_ϕ the mean curvature function of $\phi \in I_\Gamma(M, \mathbb{R}^3)$ and by $G_\phi = \epsilon\, \text{det}(-dN_\phi)$ the Gaussian curvature of $(M, \phi^* g)$.
There exists a relation between them, similar as the classical one for surfaces immersed in $\mathbb{R}^3$:

$$\mid dN_\phi \mid^2 = 4H_\phi^2 - 2\epsilon G_\phi$$

If $\epsilon = -1 \longrightarrow$ we can use Gauss-Bonnet.
If $\epsilon = 1 \longrightarrow$ we can use the following version of Gauss-Bonnet:

$$\int_M G_\phi dA_\phi + \int_{\partial M} \kappa_\phi ds = 2\Pi\chi(M),$$

where κ_ϕ is a geodesic curvature defined for curves in lorentzian surfaces.

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$:

Its conformal invariance

Using Gauss-Bonnet, the relation between the curvatures and the boundary conditions:

1. The case where $\partial M = \emptyset$ and M is compact can be regarded as a particular one

$$\boxed{\begin{array}{c}\phi \text{ critical point of } \mathcal{S} : I^{\epsilon}(M, \mathbb{R}^3) \to \mathbb{R} \Leftrightarrow \\ \phi \text{ critical point of } \mathcal{W} : I^{\epsilon}(M, \mathbb{R}^3) \to \mathbb{R}, \\ \mathcal{W}(\phi) = \int_M H_\phi^2 dA_\phi\end{array}}$$

2. When M is compact and not boundary free, and for any (Γ, N_o)

$$\boxed{\begin{array}{c}\phi \text{ critical point of } \mathcal{S} : I_\Gamma^{\epsilon}(M, \mathbb{R}^3) \to \mathbb{R} \Leftrightarrow \\ \phi \text{ critical point of } \mathcal{W} : I_\Gamma^{\epsilon}(M, \mathbb{R}^3) \to \mathbb{R}, \\ \mathcal{W}(\phi) = \int_M H_\phi^2 dA_\phi + \int_{\phi(\partial M)} \kappa^\phi ds\end{array}}$$

It is know that there aren´t solutions of the boundary free case foliated by parallel circles.

Let study the case with boundary.

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$:

Solitons foliated by parallel circles

Are there any solitons in the $O_1(3)\ NSM_2$ foliated by parallel circles?

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$:

Solitons foliated by parallel circles

Are there any solitons in the $O_1(3)\ NSM_2$ foliated by parallel circles?
We answer this question in the afirmative by showing the moduli space of solitons which are revolution surfaces around the x_3-axis.

Theorem 4 *The solitons in the $O_1(3)$ $\mathbf{NSM}_2$ (with boundary) that are invariant under rotations around the x_3-axis, correspond with the revolution surfaces around the x_3-axis generated by clamped space-like elastic curves in the Anti de-Sitter half-plane .*

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$:

Solitons foliated by parallel circles

Are there any solitons in the $O_1(3)\ NSM_{(1+1)}$ foliated by parallel circles?

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$:

Solitons foliated by parallel circles

Are there any solitons in the $O_1(3)\ NSM_{(1+1)}$ foliated by parallel circles?
We answer this question in the afirmative by showing the moduli space of solitons which are revolution surfaces around the x_3-axis.

Theorem 5 *The solitons in the $O_1(3)$ $\mathbf{NSM}_{(1+1)}$ (with boundary) invariant under rotations around the x_3-axis, correspond with the revolution surfaces around the x_3-axis generated by clamped time-like elastic curves in the Anti de-Sitter half-plane* .

$O_1(3)\ NSM_2$ & $O_1(3)\ NSM_{(1+1)}$:

In progress

- Solitons invariant under pure space-like rotations with axis not equal to the x_3-axis?

$O_1(3)$ NSM_2 & $O_1(3)$ $NSM_{(1+1)}$:

In progress

- Solitons invariant under pure space-like rotations with axis not equal to the x_3-axis?
- Solitons invariant under other subgroups of $O_1(3)$?

The end