Lorentzian Sasakian manifolds with constant pointwise ϕ-sectional curvature.

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1. Preliminaries.
   - Generalized space forms
2. Purpose
   - How?
3. Results
   - Other complex space forms
   - Lorentzian almost contact space forms
   - Lorentzian para contact space forms
   - Hyperbolic almost contact space forms

(M^{2n}, J, g) almost Hermitian:

$$J^2 = -Id$$

$$g(JX, JY) = g(X, Y)$$

Kaehlerian: $\nabla J = 0$.

$(M^{2n+1}, \phi, \xi, \eta, g)$ almost contact metric.

$$\phi^2 X = -X + \eta(X)\xi \qquad \eta(\xi) = 1$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad \eta(X) = g(X, \xi)$$

Contact metric: $\Phi = d\eta,$ with $\Phi(X, Y) = g(X, \phi Y)$.

K-contact: contact and ξ Killing.

Sasakian: $(\nabla_X \phi)Y = g(X, \phi Y) - \eta(Y)X$.

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Sasakian: $(\nabla_X \phi)Y = g(X, \phi Y) - \eta(Y)X$.

(M^n, g) **r.s.f.**

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

(N^{2n}, J, g) **c.s.f.** $N(c)$

$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\} +$$

$$+\frac{c}{4}\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}$$

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(N^{2n}, J, g) **g.c.s.f.** $N(F_1, F_2)$

$$R(X,Y)Z = F_1\{g(Y,Z)X - g(X,Z)Y\} +$$
$$+F_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}$$

$(M^{2n+1}, \phi, \xi, \eta, g)$ **S.s.f.** $M(c)$

$$
\begin{aligned}
R(X,Y)Z \;=\; & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}+\\
& +\frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+\\
& +\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}
\end{aligned}
$$

$(M^{2n+1}, \phi, \xi, \eta, g)$ **S.s.f.** $M(c)$

$$R(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$

Definition

$(M^{2n+1}, \phi, \xi, \eta, g)$ **g.S.s.f.** $M(f_1, f_2, f_3)$

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$

Study the curvature tensor of manifolds with

- Lorentzian metric
- any "contact" structure
- pointwise constant sectional curvature
- other semi-defined metric (ϕ, ξ, η).

Problem

Let be $M(\phi, \eta, \xi, g)$, g Lorentz.

$$g(\phi X, \phi X) = g(X, Y) - \eta(X)\eta(X)$$

If $X \perp \xi$ is space like $\Rightarrow$ ϕX is space like.
If $X \perp \xi$ is temporal $\Rightarrow$ ϕX is temporal.

- Other structures.
- Other relations with the metric.

Warped product

(N, J, G) almost Hermitian $\qquad M = \mathbf{R} \times_f N \qquad f > 0$

$$g_f = \pi^*(g_{\mathbf{R}}) + (f \circ \pi)^2 \sigma^*(G)$$

$$\phi(X) = (J\sigma_* X)^* \qquad \xi = \frac{\partial}{\partial t} \qquad \eta(X) = g_f(X, \xi)$$

$\Rightarrow (M, \phi, \eta, \xi, g_f)$ is almost contact metric.

Warped product

Theorem

Given $N^{2n}(F_1, F_2)$, the warped product $M^{2n+1} = \mathbf{R} \times_f N$ is a $M(f_1, f_2, f_3)$, with the following functions:

$$f_1 = \frac{(F_1 \circ \pi) - f'^2}{f^2},$$

$$f_2 = \frac{(F_2 \circ \pi)}{f^2},$$

$$f_3 = \frac{(F_1 \circ \pi) - f'^2}{f^2} + \frac{f''}{f}.$$

$$N(c) \quad \Rightarrow \quad M\left(\frac{c - 4f'^2}{4f^2}, \frac{c}{4f^2}, \frac{c - 4f'^2}{4f^2} + \frac{f''}{f}\right)$$

Metric ($J^4 = 1$)-manifold (Gadea and Montesinos)

(M^n, g) pseudo Riemannian manifold
J a $(1,1)$-tensor such that $J^4 = 1$
g and J are related by one of the following:

- $g(JX, Y) + g(X, JY) = 0$ (adapted in the electromagnetic sense metric)
- $g(JX, JY) = g(X, Y)$ (adapted Riemannian metric)

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1) Almost Hermitian manifold, $J^2 = -1$.
2) Almost para-Hermitian manifold, $J^2 = 1$ and aem.
3) Riemannian almost product manifold, $J^2 = 1$ and arm.

1) Lorentzian almost contact manifolds

$(M^{2n+1}\phi, \xi, \eta, g)$ Lorentzian almost contact manifold:

$$\phi^2 X = -X + \eta(X)\xi \qquad \eta(\xi) = 1$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \qquad \eta(X) = -g(X, \xi)$$

1) Lorentzian almost contact manifolds

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Lorentzian Sasakian: $(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X$.

Warped product

(N, J, G) almost Hermitian $\qquad M = \mathbf{R} \times_f N \qquad f > 0$

$$g_f = -\pi^*(g_{\mathbf{R}}) + (f \circ \pi)^2 \sigma^*(G)$$

$$\phi(X) = (J\sigma_* X)^* \qquad \xi = \frac{\partial}{\partial t} \qquad \eta(X) = -g_f(X, \xi)$$

$\Rightarrow (M, \phi, \eta, \xi, g_f)$ is a Lorentzian almost contact manifold.

Warped product

Theorem

Given $N^{2n}(F_1, F_2)$, the warped product $M^{2n+1} = \mathbf{R} \times_f N$ has the following curvature tensor

$$R = f_1 R_1 + f_2 R_2 + f_3 \widetilde{R}_3,$$

with

$$f_1 = \frac{(F_1 \circ \pi) + f'^2}{f^2}, \quad f_2 = \frac{(F_2 \circ \pi)}{f^2}, \quad f_3 = \frac{(F_1 \circ \pi) + f'^2}{f^2} - \frac{f''}{f}$$

and $\widetilde{R}_3(X, Y)Z =$
$= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\xi + g(Y, Z)\eta(X)\xi.$

2) Lorentzian Para-Sasakian

$(M^{2n+1}, \phi, \xi, \eta, g)$ Lorentzian almost para contact metric.

$$\phi^2 X = X + \eta(X)\xi \qquad \eta(\xi) = -1$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \qquad \eta(X) = g(X, \xi)$$

2) Lorentzian Para-Sasakian

$(M^{2n+1}, \phi, \xi, \eta, g)$ Lorentzian almost para contact metric.

$$\phi^2 X = X + \eta(X)\xi \qquad \eta(\xi) = -1$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \qquad \eta(X) = g(X, \xi)$$

L.P.-Sasakian: $(\nabla_X \phi)Y = g(X, Y) + \eta(Y)X + 2\eta(X)\eta(Y)\xi$.

Warped product

(N, J, G) almost product manifold $\qquad M = \mathbf{R} \times_f N \qquad f > 0$

$$g_f = -\pi^*(g_{\mathbf{R}}) + (f \circ \pi)^2 \sigma^*(G)$$

$$\phi(X) = (J\sigma_* X)^* \qquad \xi = \frac{\partial}{\partial t} \qquad \eta(X) = g_f(X, \xi)$$

$\Rightarrow (M, \phi, \eta, \xi, g_f)$ is a Lorentzian almost para contact manifold.

Warped product

Theorem

Let be N^{2n} an almost product manifold with

$$R(X,Y)Z = F_1\{g(V,W)U - g(U,W)V\} + F_2\{g(U,JW)JV - g(V,JW)JU\}.$$

Then $M^{2n+1} = \mathbf{R} \times_f N$ has the following curvature tensor

$$R = f_1 R_1 + f_2 \widetilde{R}_2 + f_3 R_3,$$

with

$$f_1 = \frac{(F_1 \circ \pi) - f'^2}{f^2}, \quad f_2 = \frac{(F_2 \circ \pi)}{f^2}, \quad f_3 = -\frac{(F_1 \circ \pi) - f'^2}{f^2} + \frac{f''}{f}$$

and $\widetilde{R}_2(X,Y)Z = g_f(X,\phi Z)\phi Y - g_f(Y,\phi Z)\phi X$.

3) Hyperbolic almost contact

$(M^{2n+1}, \phi, \xi, \eta, g)$ hyperbolic almost contact manifold.

$$\phi^2 X = X + \eta(X)\xi \qquad \eta(\xi) = -1$$
$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \qquad \eta(X) = g(X, \xi)$$

g is a semi-defined metric.

Warped product

(N, J, G) almost para-Hermitian manifold $\qquad M = \mathbf{R} \times_f N \qquad f > 0$

$$g_f = -\pi^*(g_{\mathbf{R}}) + (f \circ \pi)^2 \sigma^*(G)$$

$$\phi(X) = (J\sigma_* X)^* \qquad \xi = \frac{\partial}{\partial t} \qquad \eta(X) = g_f(X, \xi)$$

$\Rightarrow (M, \phi, \xi, \eta, g_f)$ is a hyperbolic almost contact manifold.

Warped product

Theorem

Given N^{2n} para-Hermitian with constant J-sectional curvature, c, the warped product $M^{2n+1} = \mathbf{R} \times_f N$ has the following curvature tensor

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3,$$

with

$$f_1 = \frac{c + 4f'^2}{4f^2},$$
$$f_2 = -\frac{c}{4f^2},$$
$$f_3 = -\frac{c + 4f'^2}{4f^2} + \frac{f''}{f}.$$

- Obstructions for some dimensions
- Structures
- Other examples

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