A survey: stability and boundedness of Volterra difference equations

V.B. Kolmanovskii, E. Castellanos-Velasco*, J.A. Torres-Muñoz

CINVESTAV-IPN, Automatic Control AV. IPN 2508 Col. Zacatenco, México D.F.
P.O. Box 14-740, MEXICO CP 07360, Mexico

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Abstract

Volterra difference equations arise in the modeling of many real phenomena. In this survey stability and boundedness problems of some Volterra nonlinear difference equations are investigated. Stability conditions and boundedness are formulated in terms of the characteristics of equations.

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1. Introduction

Volterra difference equations represent the powerful tool to solve equations with continuous time, e.g., ordinary and partial differential equations, integral and integro-differential equations, retarded and neutral type equations.

Moreover, Volterra systems describe processes whose current state is determined by their entire prehistory. These processes are encountered in the theory of viscoelasticity [16,58–60], models of propagation of perturbations in matter with memory [43,45,66,67], description of the motion of bodies with reference to hereditary [4,6,21,22,61–63], various problems of biomechanics [2,8,9,17,20], numerical solution of various type of equations with continuous time [47,53,65] and also to solve optimal control problems [41,42,45].

* Corresponding author.

E-mail address: ecastell@ctrl.cinvestav.mx (E. Castellanos-Velasco).
The aim of this survey is to develop a technique for investigating stability and boundedness of the discrete Volterra difference equations. Such a theory is largely nonexistent at this time and should be developed before any significant progress can be made in the theory for Volterra equations. Previously only a few papers deal with the theory of general Volterra equations and most of them are devoted to the stability analysis of Volterra linear difference equations with constant coefficients or of convolution type.

For example, in [18] the asymptotic behaviour of the solution is established for renewal equations which are linear difference Volterra equations with nonnegative coefficients. In the book [68], the stability conditions are shown for the class of \((\rho, \sigma)\) linear multistep methods applied to Volterra convolution integral and integro-differential equations obtained by using the discrete version of Wiener–Paley theorem for the convolution difference Volterra equations. In [60], stability criteria for difference equations of Volterra-type with degenerate kernel are derived.

During the last few years, stability of Volterra difference equations has been investigated in a number of papers. In particular various methods to investigate stability of Volterra difference equations were used such as direct Lyapunov method [19,37,55,56], comparison theorems [51,57], \(Z\)-transform, topological methods, etc.

For finite order equations, this method allow us to formulate stability conditions in terms of existence of Lyapunov functions which depend on finite number of arguments [36,38,39,52,69]. In [48,50], similar results were proved for difference equations of unbounded order and, by the aid of suitable Lyapunov functions, some stability criteria for the class of linear multistep methods for Volterra integro-differential equations were derived.

Applications of Lyapunov direct method and some stability conditions for Volterra discrete equations were obtained in [28–32]. In connection with these papers, it appeared clear that the problem arises to create some formal procedure which would lead to explicit form of several Lyapunov functions and as a consequence of this to the corresponding stability conditions for the discrete equations under consideration. Once a list of stability conditions is obtained, we can choose the most useful for these difference equations.

The results of [10–15] naturally stimulate the elaboration of a formal procedure that would aid in deriving the functionals in an explicit form and, consequently, the stability conditions for difference equations. A similar question for heredity and continuous time equations is examined in [10]. The procedure designed in [10] is elegant in that the unknown functional is derived via certain transformations of the Lyapunov function constructed for a special auxiliary ordinary equation. The underlying principle of the procedure developed for Volterra’s equations also consists in the use of the Lyapunov functions constructed for auxiliary difference equations that depend on an a priori defined finite number of steps. Unlike in the case of continuous time equations, the number of steps here is an additional variable parameter which can be used in every application of the procedure.

In this form, a natural question arises about the ways of constructing the Lyapunov functional \(V\). The essence of the procedure proposed is to construct Lyapunov functions for some auxiliary specially constructed difference equations depending on finite number of steps instead of doing it directly for the unbounded order difference equation. This
number of steps is an additional parameter that must be chosen at the process of the procedure application.

The conditions of boundedness of the process are of great interest in these applications. For general discrete time Volterra systems, the boundedness conditions are formulated sometimes through Lyapunov’s second method. In the book [23,24], topological methods were used to study stability in the first approximation of some nonlinear Volterra difference equations. The same approach will be used for estimation of the limitedness and boundedness on the solution. The knowledge of these bounds is quite important because they represent the error between exact solution of the original problem and its difference approximation.

To obtain these bounds we shall interpret Volterra equations as operator equations in appropriate spaces [33,53,55]. Such approach for integral equations was used in the books [7–9] and for functional differential equation in the books [3,45,62]. But owing to the difficulties in constructing the Lyapunov functions, concrete systems may be studied by methods that take into account the specifics of these systems, e.g., comparison principle, stability in the first approximation [44], necessary and sufficient conditions for exponential stability, topological theorems, etc. In the stochastic systems like random processes, the expectation let modify the domain of stability for obtaining different conditions [33,49,52,54].

The structure of this survey is the following. In Section 2, we give representation of the resolvent and their solutions for implicit and explicit schemes, which will be used later. Section 3 is devoted to the stability conditions for linear and nonlinear equations, formulated immediately in terms of the coefficients. In Section 4, we derive boundedness conditions in various sense for Volterra difference equations.

2. Resolvent

In this section, we consider the form of equations defining the resolvent, and also the general representation of the solution for some Volterra discrete equations.

2.1. Implicit schemes

Consider a system of linear equations:

\[ x_n = \sum_{k=0}^{n} a_{n,k} x_k + f_n, \quad n \geq 0, \tag{2.1} \]

where \( n \) and \( k \) are integers, vectors \( x_n \in \mathbb{R}^r \), \( R^r \) is a linear \( r \)-dimension space equipped with some norm \( |\cdot| \), \( a_{n,k} \) are prescribed \( r \times r \) matrices, and finally \( f_n \in \mathbb{R}^r \) are a given sequence of perturbations.

Let us assume that a unique solution \( x_n \) of system (2.1) does exist for all finite \( n \). Note that a sufficient condition for the solution existence and unicity is the following: \( \det(I - a_{n,n}) \neq 0 \) for all \( n \geq 0 \). Also, if the solution of system (2.1) does exist for arbitrary \( f_n \), then the conditions \( \det(I - a_{n,n}) \neq 0 \), for \( n \geq 0 \) become necessary as well.

Let us find the solution \( x_n \) as a function of \( f_k \), \( k \leq n \) and auxiliary \( r \times r \) matrix \( R_{m,n}, 0 \leq n \leq m \), referred to as a resolvent. Multiplying both sides of equation (2.1)
by $R_{m,n}$ from the left and summing with respect to $n$ from $n = 0$ to $n = m$, we obtain

$$\sum_{n=0}^{m} R_{m,n}(x_n - f_n) = \sum_{n=0}^{m} R_{m,n} \sum_{k=0}^{n} a_{n,k} x_k$$

$$= \sum_{k=0}^{m} \sum_{n=k}^{m} R_{m,n} a_{n,k} x_k = \sum_{n=0}^{m} \sum_{k=n}^{m} R_{m,k} a_{k,n} x_n.$$  \hfill (2.2)

From equality (2.2) it follows that

$$\sum_{n=0}^{m} \left[ R_{m,n}(x_n - f_n) - \sum_{k=n}^{m} R_{m,k} a_{k,n} x_n \right] = 0,$$

$$\sum_{n=0}^{m} \left[ R_{m,n}(x_n - f_n) - \sum_{k=n}^{m} (R_{m,k} a_{k,n} \pm a_{m,n}) x_n \right] = 0.$$

Let us require that the resolvent $R_{m,n}$ satisfies, for any $m$ and $0 \leq n \leq m$, the next relation

$$R_{m,n} = \sum_{k=n}^{m} R_{m,k} a_{k,n} - a_{m,n}, \quad 0 \leq n \leq m, \ m\text{-fixed.}$$  \hfill (2.3)

Then, by virtue of (2.2) and (2.3)

$$\sum_{n=0}^{m} R_{m,n}(x_n - f_n) = \sum_{n=0}^{m} (R_{m,n} + a_{m,n}) x_n,$$

and hence

$$\sum_{n=0}^{m} (R_{m,n} f_n + a_{m,n} x_n) = 0.$$  \hfill (2.4)

Changing the sum in the right hand side of (2.2) according to (2.4) yields the desired form of the solution

$$x_n = f_n - \sum_{k=0}^{n} R_{n,k} f_k.$$  \hfill (2.5)

So, if the solution of equations (2.1) and (2.3) do exist, then it can be represented in the form (2.5).

Taking the following fact:

$$\sum_{l=k}^{n} R_{n,l} a_{l,k} = \sum_{l=k}^{n} a_{n,l} R_{l,k}, \quad l \leq k \leq n.$$  \hfill (2.6)
Thus with the help of (2.3) and (2.6) we have

\[
fn - \sum_{k=0}^{n} R_{n,k} f_k = fn - \sum_{l=0}^{n} a_{n,l} \sum_{k=0}^{l} R_{l,k} f_k + \sum_{k=0}^{n} a_{n,k} f_k.
\]  

(2.7)

Substituting \( x_n \) from (2.5) into (2.7), we have

\[
\sum_{k=0}^{n} (R_{n,k} + a_{n,k}) f_k = \sum_{l=0}^{n} a_{n,l} \sum_{k=0}^{l} R_{l,k} f_k = \sum_{k=0}^{n} \sum_{l=k}^{n} a_{n,l} R_{l,k} f_k.
\]

Hence:

\[
\sum_{k=0}^{n} \left[ R_{n,k} - \sum_{l=k}^{n} a_{n,l} R_{l,k} + a_{n,k} \right] f_k = 0.
\]

From here and arbitrariness of \( f_k \), since from (2.5) \( R_{l,k} \) does not depend on the \( f_k \), it follows that the resolvent satisfies also the equation:

\[
R_{n,k} = \sum_{l=k}^{n} a_{n,l} R_{l,k} + a_{n,k}, \quad k \leq n, k\text{-fixed}.
\]  

(2.8)

Comparing (2.5) and (2.8), one verifies (2.7). Note that, if the matrices \( a_{n,k} \) in Eq. (2.1) only depend on the difference \( “n − k” \) (i.e., \( a_{n,k} = a_{n−k} \) for all \( 0 \leq k \leq n \)), then the resolvent \( R_{n,k} \) also only depends on the difference \( “n − k” \) (i.e., \( R_{n,k} = R_{n−k} \)) and moreover by virtue of (2.5) and (2.8):

\[
R_n = -a_n + \sum_{k=0}^{n} a_{n−k} R_k = -a_n + \sum_{k=0}^{n} R_{n−k} a_k.
\]  

(2.9)

2.2. Nonlinear equations

Eqs. (2.5) and (2.8) for the resolvent allow the derivation of the variation of constants formula for the nonlinear Volterra difference equation (with \( G_k : R^r \rightarrow R^r \)):

\[
x_n = f_n + \sum_{k=0}^{n} a_{n,k} [x_k + G_k(x_k)].
\]  

(2.10)

Let us define the resolvent \( R_{m,n} \) by relation (2.3) (where \( 0 \leq n \leq m, m\text{-fixed} \)), or by relation (2.8) (where \( k \leq n, k\text{-fixed} \)).

Since this equation is obtained from (2.1) by replacing \( f_n \) with \( [f_n + \sum_{k=0}^{n} a_{n,k} G_k(x_k)] \), by (2.5) we have

\[
x_n - f_n + \sum_{k=0}^{n} R_{n,k} f_k - \sum_{l=0}^{n} a_{n,l} G_l(x_l) = -\sum_{k=0}^{n} \sum_{l=0}^{k} R_{n,k} a_{k,l} G_l(x_l)
\]

\[
= -\sum_{l=0}^{n} \sum_{k=l}^{n} R_{n,k} a_{k,l} G_l(x_l).
\]
Consequently
\[
x_n - f_n + \sum_{k=0}^{n} R_{n,k} f_k + \sum_{l=0}^{n} \left[-a_{n,l} + \sum_{k=l}^{n} R_{n,k} a_{k,l} \right] G_l(x_l) = x_n - f_n + \sum_{k=0}^{n} R_{n,k} f_k + \sum_{k=0}^{n} R_{n,k} G_k(x_k) = 0.
\]

From here it follows the variation of constants formula
\[
x_n = f_n - \sum_{k=0}^{n} R_{n,k} (f_k + G_k(x_k)), \quad n \geq 0. \tag{2.11}
\]

2.3. Explicit schemes

In a similar way, we can obtain an expression for the solution of the Volterra equation:
\[
x_{n+1} = a_n x_n + \sum_{k=s}^{n} a_{n,k} x_k + f_n, \quad n \geq s, \quad x_s = x_0. \tag{2.12}
\]

Here \( s \) is an initial time moment, \( x_0 \) is a prescribed initial condition, \( a_n \) is a given sequence of \( r \times r \) matrices, \( f_n \) are perturbations and all other notations are the same as in Eq. (2.1). Let us denote by \( R_{m,n} \) the resolvent of equation which represents the \( r \times r \) matrix with \( R_{m,m} = I \), where \( I \) is an identity matrix.

Multiplying both sides of Eq. (2.12) by \( R_{m+1,n+1} \) from the left and summing in \( n \) from \( n = s \) to \( m \geq s \), we obtain that
\[
\sum_{n=s}^{m} R_{m+1,n+1} x_{n+1} = x_{m+1} - R_{m+1,s} x_0 + \sum_{n=s}^{m} R_{m+1,n} x_n
\]
\[
= \sum_{n=s}^{m} \left[ \left( R_{m+1,n+1} a_n + \sum_{k=n}^{m} R_{m+1,k+1} a_{k,n} \right) x_n + R_{m+1,n+1} f_n \right].
\]

Hence, the resolvent \( R_{m,n} \) for any fixed \( m \) satisfies (as a function of \( n \leq m \)) the equation:
\[
R_{m+1,n} = R_{m+1,n+1} a_n + \sum_{k=n}^{m} R_{m+1,k+1} a_{k,n}, \quad n \leq m, \quad R_{m+1,m+1} = I, \quad m\text{-fixed}.
\]

In addition
\[
x_{m+1} = R_{m+1,s} x_s + \sum_{n=s}^{m} R_{m+1,n+1} f_n. \tag{2.13}
\]

Let us assume that the vector \( f_n = 0 \) in (2.13) and that the vector \( x_s \) has its \( \ell \)th component equal to 1 with all other components of \( x_s \) equal to zero. Then, by virtue of (2.13), the \( \ell \)th column of the resolvent satisfies Eq. (2.12) for \( m \geq s \). Giving to \( \ell \)
the values from 1 to \( n \), we can conclude that the resolvent \( R_{m,n} \) (as a function of \( m \) for any fixed \( n \)) satisfies the following homogeneous equation:

\[
R_{m+1,n} = a_m R_{m,n} + \sum_{k=n}^{m} a_{m,k} R_{k,n}, \quad m \geq n, \quad R_{n,n} = I, \ j \text{-fixed}, \ m \geq n.
\]

If \( R_{n,n} \) depends only on the difference of arguments \( m-n \) then we have the following equations for resolvent:

\[
R_{m+1} = a R_m + \sum_{k=n}^{m} a_{m-k} R_k = a R_m + \sum_{k=n}^{m} R_{m-k} a_k,
\]

and Eq. (2.13) is represented as follows:

\[
x_{m+1} = R_{m+1} x_s + \sum_{j=s}^{m} R_{m-j} f_j, \quad x_s = x_0.
\]

3. Stability of the solutions for Volterra difference equations

3.1. Statement of the problem

Let \( S \) be a space of sequences whose elements belong to \( \mathbb{R}^r \) and let \( f : N \times S \rightarrow \mathbb{R}^r \) be a functional such that \( f(n,y_0,y_1,\ldots,y_n,\ldots) \) does not depend on \( y_j \) for \( j > n \) and \( f(n,0,\ldots,0) = 0 \), for all \( n \). Let us consider the following general Volterra difference equations:

\[
y_{n+1} = F(n,y_{n-1},\ldots,y_n,f(n,y_0,\ldots,y_n)), \quad n = 0,1,\ldots, \quad y_n \in \mathbb{R}^r,
\]

where: \( F: N \times \mathbb{R}^{(l+2)r} \rightarrow \mathbb{R}^r \) and \( l \) is a given nonnegative integer. It is supposed that the solution of Eq. (3.1) exists for all initial conditions belonging to some domain \( D \) containing the origin.

Initial conditions for Eq. (3.1) are given by the set of vectors \( \{y_{-l},\ldots,y_0\} \). If initial conditions are zero, then (3.1) has the zero solution. We should like to investigate stability of the zero solution with respect to the disturbances of initial conditions.

**Definition 3.1.** The zero solution of (3.1) is said to be:

1. stable if for all \( \varepsilon > 0 \), there exists a \( \delta = \delta(\varepsilon) > 0 \) such that \( \|y_n\| < \varepsilon, \ n \geq 0 \) if \( \|y_j\| < \delta(\varepsilon), \ -l \leq j \leq 0; \)
2. asymptotically stable if it is stable and \( \lim_{n \to \infty} y_n = 0 \), for all \( (y_{-l},\ldots,y_0) \) from some domain \( Q \subset \mathbb{R}^{(l+1)r} \) which is named as an attraction domain of the zero solution;
3. stable in the whole if it is asymptotically stable and \( \lim_{n \to \infty} x(n,0,x_0) = 0, \ x_0 \in \mathbb{R}^r \).

Remark that if the zero solution is stable, then the solution of (3.1) exists for all initial conditions belonging to some domain \( D \).
Stability conditions are formulated in terms of the functional $V : \mathbb{N} \times \mathbb{S} \to \mathbb{R}$ such that $V(n, y_0, \ldots, y_n, \ldots) \equiv V(n, y_0, \ldots, y_n)$ and scalar continuous nondecreasing functions $\omega_j : \mathbb{R}^+ \to \mathbb{R}^+$ and $\omega_j(0) = 0$.

The reader is referred to [25] for a comprehensive coverage of the subject; nevertheless here, the following theorem will be useful in the sequel.

**Theorem 3.1.** Assume that in some domain \( \{(n \geq 0) \times (\|y_n\| \leq k)\} \subset \mathbb{N} \times \mathbb{S}, \) for some $k > 0$ there exists a functional $V$ and the functions $\omega_1 - \omega_3$ such that for any $n \geq 0$, the following relations are valid:

1. $\omega_1(\|y_n\|) \leq V(n, y_{-1}, \ldots, y_n)$;
2. $V(0, y_{-1}, \ldots, y_0) \leq \omega_2(\max\{\|y_{-1}\|, \ldots, \|y_0\|\})$;
3. $\Delta V = V(n+1, y_{-1}, \ldots, y_n, y_{n+1}) - V(n, y_{-1}, \ldots, y_n) \leq -\omega_3(\|y_n\|)$ (or $\leq 0$), then the zero solution of (3.1) is asymptotically stable (or stable).

**Remark 3.1.** Some assumptions of Theorem 3.1 can be weakened. Thus, for example, instead of the estimate from below in condition 1 of Theorem 3.1 (i.e., the requirement of positive definiteness of the functional $V$) it is sufficient to assume that

4. $V(n, y_{-1}, \ldots, y_n) \geq \omega_1(\|y_n + F_3(n, y_{-1}, \ldots, y_{n-1})\|)$, \quad (3.2)

where $F_3$ is a functional $F_3 : \mathbb{N} \times \mathbb{S} \to \mathbb{R}$, $F_3(n, 0, \ldots, 0)$ such that the zero solution of the equation:

$$y_n + F_3(n, y_{-1}, \ldots, y_{n-1}) = 0,$$ \quad (3.3)

is asymptotically stable. This hypothesis can be useful because in some cases the stability investigation of the latter difference equations is simpler than the original one.

**3.1.1. Description of the procedure**

In connection with Theorem 3.1, a natural question arises about the ways of constructing the functional $V$. The essence of the procedure proposed below is to construct Lyapunov functions for some auxiliary specially constructed difference equations depending on finite number of steps instead of doing it directly for the unbounded order difference equation (3.1).

This number of steps is an additional parameter that must be chosen at the process of the procedure application.

The proposed procedure consists of the following steps.

(A) Right hand side $F$ of Eq. (3.1) must be represented either in the form:

$$F = F_1(n, y_{n-\tau}, y_{n-\tau+1}, \ldots, y_n) + F_2(n, y_{-1}, y_{-1+1}, \ldots, y_n),$$

$$F_1(n, 0, \ldots, 0) = 0, \quad F_2(n, 0, \ldots, 0) = 0,$$ \quad (3.4)

or in the form

$$F = F_1(n, y_{n-\tau}, y_{n-\tau+1}, \ldots, y_n) + \Delta F_2(n, y_{-1}, y_{-1+1}, \ldots, y_n),$$ \quad (3.5)

where: $\tau \geq 0$ is any fixed integer for which Volterra equation can be chosen arbitrary and the operator $\Delta$ is defined by condition 3 of Theorem 3.1.
(B) For auxiliary difference equation of order \( \tau + 1 \):

\[ x_{n+1} = F_1(n, x_{n-\tau}, x_{n-\tau+1}, \ldots, x_n), \]  

one must construct Lyapunov functional \( v(n, x_{n-\tau}, \ldots, x_n) \) satisfying conditions of some stability theorems for finite order equations, e.g., satisfying the estimates

\[ \omega_4(\|x_n\|) \leq v(n, x_{n-\tau}, \ldots, x_n), \]

\[ v(0, x_{-\tau}, \ldots, x_0) \leq \omega_5(\max\{\|x_{-\tau}\|, \ldots, \|x_0\|\}). \]  

(3.7)

\[ \Delta v = v(n + 1, x_{n+1-\tau}, \ldots, x_n, F_1(n, x_{n-\tau}, \ldots, x_n)) - v(n, x_{n-\tau}, \ldots, x_n) \leq \omega_6(\|x_n\|). \]

(C) Represent the required functional \( V \) as a sum \( V = V_1 + V_2 \) where

\[ V_1(n, y_{-\tau}, \ldots, y_n) = v(n, y_{n-\tau}, \ldots, y_n), \]  

if representation (3.4) is used, or

\[ V_1(n, y_{-\tau}, \ldots, y_n) = V(n, y_{n-\tau}, \ldots, y_{n-1}, y_n - F_2(n, y_{-\tau}, \ldots, y_n)), \]  

if representation (3.5) is used.

(D) Choose \( V_2(n, y_{-\tau}, \ldots, y_n) \) in such a way that the functional \( V = V_1 + V_2 \) will satisfy the hypotheses of Theorem 3.1 or remark to it.

Remark 3.2. We want to stress that the construction of the functional \( V_1 \) is a problem easier than the direct construction of the whole functional \( V \), because it is related to a finite order difference equation. Moreover, the construction of the second part \( V_2 \) of the functional derives almost automatically from the choice of \( V_1 \). For these reasons \( V_1 \) will be referred to as a main component of the required functional \( V \).

Remark 3.3. The realization of the various steps of the procedure (i.e., the choice of the number \( \tau \), the splitting of the functional \( F \), and the construction of Lyapunov functional \( v \)) can be fulfilled nonuniquely. This nonuniqueness has to be essentially used, because, depending on various resulting functionals \( V \), we could obtain different stability conditions and also different parts of the stability domain of the considered difference equation.

Remark 3.4. If representation (3.5) is used, the resulting functional \( V \) could appear to be only nonnegative definite with respect to the variables \( y_n \); in other words, since from (3.9), \( V_1 \) depends not only on \( y_n \), but also on \( F_2 \), it may happen that \( V \) does not easily satisfies assumption 3 of Theorem 3.1. In such a case, condition 4 of Remark 3.1 with \( F_3 = -F_2 \) should be used and thus, from (3.3) additional investigation about stability of the zero solution of equation

\[ y_n + F_2(n, y_{-\tau}, \ldots, y_{n-1}) = 0 \]  

(3.10)

is required.
3.2. Linear equations of convolution type

Let us consider the scalar system

\[ x_{n+1} = - \sum_{i=0}^{n} a_{n-i} x_i, \quad n \geq 0, \]  

(3.11)

where \( a_n \) is the given sequence of coefficients. The initial position \( x_0 \) of the system \( (3.11) \) is given for \( n = 0 \). To study the properties of system \( (3.11) \), we define on its solutions the functional \( V_0 = V_0(n, x_0, \ldots, x_n) \) equal to

\[ V_0 = (2 - a_{-1}) x_n^2 + a_{n+1} \left( \sum_{i=0}^{n} x_i \right)^2 - \sum_{i=0}^{n} (a_{n+1-i} - a_{n-i}) \left( \sum_{j=i}^{n} x_j \right)^2, \]  

(3.12)

where \( a_{-1} \) is a constant which is chosen below.

We calculate the first-order difference \( \Delta V_0 = V_0(n+1, x_0, \ldots, x_{n+1}) - V_0(n, x_0, \ldots, x_n) \) of functional \( (3.12) \) by substituting successively the solutions of system \( (3.11) \). For the first addend in the right hand side of \( (3.12) \), we get:

\[ \Delta [(2 - a_{-1}) x_n^2] = (2 - a_{-1}) (x_{n+1}^2 - x_n^2) \]

\[ = -(2 - a_{-1}) x_{n+1} \left( \sum_{i=0}^{n} a_i x_{n-i} \right) - (2 - a_{-1}) x_n^2 \]

\[ = -2 x_{n+1} \sum_{i=0}^{n} a_i x_{n-i} - a_{-1} x_{n+1}^2 - (2 - a_{-1}) x_n^2. \]

We denote

\[ G_1 = a_{n+1} \left( \sum_{i=0}^{n} x_i \right)^2, \quad G_2 = - \sum_{i=0}^{n} (a_{n+1-i} - a_{n-i}) \left( \sum_{j=i}^{n} x_j \right)^2. \]

Then

\[ \Delta G_1 = a_{n+2} \left( \sum_{i=0}^{n+1} x_i \right)^2 - a_{n+1} \left( \sum_{i=0}^{n} x_i \right)^2. \]

But

\[ -a_{n+1} \left[ \sum_{i=0}^{n} x_i \pm x_{n+1} \right]^2 = -a_{n+1} \left[ \sum_{i=0}^{n+1} x_i - x_{n+1} \right]^2 \]

\[ = -a_{n+1} \left( \sum_{i=0}^{n+1} x_i \right)^2 - a_{n+1} \left( x_{n+1}^2 - 2 x_{n+1} \sum_{i=0}^{n+1} x_i \right). \]
\[ = a_{n+1} \left[ x_{n+1}^2 - 2x_{n+1} \left( \sum_{i=0}^{n} x_i + x_{n+1} \right) \right] \]

\[ = 2a_{n+1}x_{n+1} \sum_{i=0}^{n} x_i + a_{n+1}x_{n+1}^2. \]

Consequently

\[ \Delta G_1 = (a_{n+2} - a_{n+1}) \left( \sum_{i=0}^{n+1} x_i \right)^2 + a_{n+1}x_{n+1} \left( x_{n+1} + 2 \sum_{i=0}^{n} x_i \right). \]

Now for \( G_2 \) we get

\[
\begin{align*}
\Delta G_2 &= - \sum_{i=0}^{n+1} (a_{n+2-i} - a_{n+1-i}) \left( \sum_{j=i}^{n+1} x_j \right)^2 + \sum_{i=0}^{n} (a_{n+1-i} - a_{n-i}) \left( \sum_{j=i}^{n} x_j \right)^2 \\
& \quad \pm \sum_{i=0}^{n+1} (a_{n+1-i} - a_{n-i}) \left( \sum_{j=i}^{n+1} x_j \right)^2 \\
& = - \sum_{i=0}^{n+1} (a_{n+2-i} - 2a_{n+1-i} + a_{n-i}) \left( \sum_{j=i}^{n+1} x_j \right)^2 + G_3.
\end{align*}
\]

Here “±” stands for “adding and subtracting” and it is assumed that

\[
G_3 = \sum_{i=0}^{n} (a_{n+1-i} - a_{n-i}) \left( \sum_{j=i}^{n} x_j \right)^2 - \sum_{i=0}^{n+1} (a_{n+1-i} - a_{n-i}) \left( \sum_{j=i}^{n+1} x_j \right)^2
\]

\[ = G_4 - x_{n+1}^2 (a_0 - a_{-1}), \]

\[
G_4 = -x_{n+1} \sum_{i=0}^{n} (a_{n+1-i} - a_{n-i}) \left( x_{n+1} + 2 \sum_{j=i}^{n} x_j \right).
\]

The functional \( G_4 \) is in turn representable as

\[
G_4 = -x_{n+1}^2(a_{n+1} - a_0) - 2x_{n+1} \sum_{i=0}^{n} x_i \sum_{j=0}^{i} (a_{n+1-j} - a_{n-j})
\]

\[ = -x_{n+1}^2(a_{n+1} - a_0) - 2a_{n+1}x_{n+1} \sum_{i=0}^{n} x_i + 2x_{n+1} \sum_{i=0}^{n} a_{n-i}x_i. \]
Finally, we get
\[
\Delta V_0 = J_0(n,x) = (a_{n+2} - a_{n+1}) \left( \sum_{i=0}^{n+1} x_i \right)^2 - \sum_{i=0}^{n+1} (a_{n+2-i} - 2a_{n+1-i} + a_{n-i}) \left( \sum_{j=i}^{n+1} x_j \right)^2 - (2 - a_{-1})x_n^2.
\]
(3.13)

The relationship (3.12) and (3.13) allow one to draw some conclusions about the stability of the solutions of Eq. (3.11).

**Theorem 3.2.** Let there be a constant \(a_{-1}\) such that the inequalities:

\[
0 \leq a_{-1} < 2, \quad a_j \geq 0, \quad a_{j+n} \leq a_j,
\]

\[
a_{j+2} - 2a_{j+1} + a_j \geq 0, \quad j \geq -1
\]

are valid. Then, any solution \(x(n,0,x_0)\) of Eq. (3.11) tends to zero as \(n \to \infty\).

**Example 3.1.** Consider a Volterra difference equation of convolution type

\[
x_{n+1} = \sum_{i=0}^{n} a_ix_{n-i}, \quad n \geq 0.
\]

According to the procedure let us take the functional in the form \(V = V_1 + V_2\). Here \(V_1 = |x_n|\). Then

\[
\Delta V = \Delta V_1 + \Delta V_2 \leq (|a_0| - 1)|x_n| + \sum_{i=1}^{n} |a_i x_{n-i}| + \Delta V_2.
\]

Take the second functional \(V_2\):

\[
V_2 = \sum_{j=1}^{n} |x_{n-j}| \sum_{k=j}^{\infty} |a_k|,
\]

then the functional

\[
V = |x_n| + \sum_{j=1}^{n} |x_{n-j}| \sum_{k=j}^{\infty} |a_k| \quad \text{and} \quad \Delta V \leq |x_n| \left( \sum_{j=0}^{\infty} |a_j| - 1 \right),
\]

satisfies all the conditions of Theorem 3.1.

**3.3. Linear nonconvolution equations**

The above approach to stability of the solutions of Eq. (3.11) can be used to consider some equations of the form

\[
x_{n+1} = -\sum_{i=0}^{n} a_{n,i} x_i, \quad n \geq 0,
\]

(3.15)

where \(a_{n,i}\) are sets of numbers defined for \(n \geq 0\) and \(0 \leq i \leq n\).
We introduce sequences \( \{a_{n-1, n} \}_{n \geq 0} \) and \( \{a_{n-1} \}_{n \geq 0} \), which satisfy the conditions

\[
0 \leq \sup_{n \geq 0} \{a_{n-1, n} \} < 2, \quad a_{n-1} \geq 0, \quad a_{n+1, -1} - a_{n-1} \leq 0
\]  

(3.16)

and consider the solutions of system (3.15) the functional \( V_0 = V_0(n, x_0, \ldots, x_n) = G_1 + G_2 + G_3 \) with

\[
G_1 = a_{n,-1} \left( \sum_{i=0}^{n} x_i \right)^2, \quad G_2 = -\sum_{i=0}^{n} (a_{n,i-1} - a_{n,i}) \left( \sum_{j=i}^{n} x_j \right)^2,
\]

\[
G_3 = (2 - a_{n-1,n})x_n^2.
\]  

(3.17)

We calculate \( \Delta G_k \) for \( k = 1, 2, 3 \) and obtain that

\[
\Delta G_1 = a_{n+1,-1} \left( \sum_{i=0}^{n+1} x_i \right)^2 - G_1 \pm a_{n,-1} \left( \sum_{i=0}^{n+1} x_i \right)^2
\]

\[
= (a_{n+1,-1} - a_{n,-1}) \left( \sum_{i=0}^{n+1} x_i \right)^2 + a_{n,-1}x_{n+1} \left( x_{n+1} + 2 \sum_{i=0}^{n} x_i \right).
\]

Next

\[
\Delta G_2 = \Delta G_2 \pm \sum_{i=0}^{n+1} (a_{n,i-1} - a_{n,i}) \left( \sum_{j=i}^{n+1} x_j \right)^2
\]

\[
= -\sum_{i=0}^{n+1} (a_{n+1,i-1} - a_{n+1,i} - a_{n,i-1} + a_{n,i}) \left( \sum_{j=i}^{n+1} x_j \right)^2 + G_4.
\]

Here

\[
G_4 = \sum_{i=0}^{n} (a_{n,i-1} - a_{n,i}) \left( \sum_{j=i}^{n} x_j \right)^2 - \sum_{i=0}^{n+1} (a_{n,i-1} - a_{n,i}) \left( \sum_{j=i}^{n+1} x_j \right)^2
\]

\[
= -x_{n+1}^2 (a_{n,n} - a_{n,n+1}) - x_{n+1} \sum_{i=0}^{n} (a_{n,i-1} - a_{n,i}) \left( x_{n+1} + 2 \sum_{j=1}^{n} x_j \right)
\]

\[
= -x_{n+1}^2 (a_{n,n+1} - a_{n,-1}) - 2x_{n+1} a_{n,-1} \sum_{i=0}^{n} x_i + 2x_{n+1} \sum_{i=0}^{n} a_{n,i} x_i.
\]

Finally

\[
\Delta G_3 = -2x_{n+1} \sum_{i=0}^{n} a_{n,i} x_i - (2 - a_{n-1,n})x_n^2 - a_{n,n+1} x_{n+1}^2.
\]
As the result, we obtain

\[ \Delta V_0 = J_2(n,x) = -(2 - a_{n-1,n})x_n^2 + (a_{n+1,-1} - a_{n,-1}) \left( \sum_{j=0}^{n+1} x_j \right)^2 \]

\[ - \sum_{i=0}^{n+1} \left( a_{n+1,i-1} - a_{n+1,i} - a_{n,i-1} + a_{n,i} \right) \left( \sum_{j=i}^{n+1} x_j \right)^2. \]  

(3.18)

Now, we formulate a counterpart of Theorem 3.2.

**Theorem 3.3.** Let conditions (3.16) be satisfied and also:

\[
a_{n,i-1} - a_{n,i} \leq 0, a_{n+1,i-1} - a_{n+1,i} - a_{n,i-1} + a_{n,i} \geq 0,
\]

\[
\inf_{n \geq 0} \{a_{n,0}\} > 0.
\]  

(3.19)

Then, the solutions of Eq. (3.15) tends to zero as \( n \to \infty \).

The proof of Theorem 3.3 follows the lines of the proof of Theorem 3.2 where the functional \( V_0 \) and its first order difference \( \Delta V_0 \) obey (3.17) and (3.18).

**Example 3.2.** Consider a nonstationary equation of nonconvolution type

\[ x_{n+1} = \sum_{i=0}^{n} a_{n,i}x_{n-i}, \quad n \geq 0. \]  

(3.20)

Let us introduce the functional \( V = V_1 + V_2 \), where

\[ \Delta V_1 = (|a_{n,0}| - 1)|x_n| + |a_{n,1}x_{n-1}| + \cdots + |a_{n,n}x_0|, \]

\[ \Delta V_2 = \sum_{k=1}^{n} |x_{n-k}| \sum_{j=k}^{\infty} |a_{n+j-k,j}|. \]

Then we obtain

\[ \Delta V \leq |x_n| \left[ \sup_{n \geq 0} \left\{ \sum_{j=0}^{\infty} |a_{n+j,j}| \right\} - 1 \right]. \]

As a result we have positive definite, decrescent functional, satisfying the condition

\[ \sup_{n \geq 0} \left\{ \sum_{j=0}^{\infty} |a_{n+j,j}| \right\} < 1. \]

It means that all the solutions of Eq. (3.20) tends to zero as \( n \to \infty \).
3.4. Multidimensional case

Consider the Volterra equation of discrete time

\[ x_{n+1} = F(n, x_0, \ldots, x_n), \quad n \geq 0, \quad x \in \mathbb{R}^r. \] (3.21)

The solution of equation is defined by the initial condition \( x_0 \in \mathbb{R}^r \).

It is assumed that \( F(n, 0, \ldots, 0) \) for all \( n \geq 0 \). Let us denote the solution of Eq. (3.21) with initial condition \( x_0 \) by \( x(n, 0, x_0) \).

Let us denote by \( \varphi_j(|x|) \) continuous increasing functions equals zero for \( x = 0 \), by \( V(n, x_0, \ldots, x_n) \) a scalar functional depending on \( n \geq 0 \) and arguments \( x_0, \ldots, x_n \) by \( I \)-identity \( r \times r \) matrix, and prime is transposition sign.

The problem of stability in the whole is connected with the problem of absolute stability in which equation of the motion are linear with constant coefficients and perturbations (as a rule nonlinear) belong to some domain. The basic methods to investigate the problem of absolute stability are frequency method and direct Lyapunov method. In the theory of absolute stability Lyapunov function is chosen as a quadratic form plus integral from nonlinear perturbation.

Lemma 3.1. Assume that the following conditions are fulfilled:

\[ \varphi_1(|x|) \leq V(n, x_0, \ldots, x_n), \] (3.22)

\[ V(0, x_0) \leq \varphi_2(|x_0|), \] (3.23)

\[ \Delta V(n, x_0, \ldots, x_n) = V(n + 1, x_0, \ldots, x_n, F(n, x_0, \ldots, x_n)) - V(n, x_0, \ldots, x_n) \leq \varphi_3(|x_n|), \] (3.24)

\[ \varphi_1(|x|) \to \infty, \quad |x| \to \infty. \] (3.25)

Then the zero solution of Eq. (3.21) is asymptotically stable in the whole.

Proof. Take any \( \varepsilon > 0 \) and choose \( \delta(\varepsilon) > 0 \) such that \( \varphi_2(\delta) \leq \varphi_1(\varepsilon) \). Assume that initial condition \( x_0 \) belongs to the ball \( |x_0| \leq \delta \). Then by virtue of assumptions (3.22)–(3.25) we have

\[ \varphi_1(|x|) \leq V(n, x_0, \ldots, x_n) \leq V(0, x_0) \leq \varphi_2(\delta) \leq \varphi_1(\varepsilon). \]

Hence \( |x| \leq \varepsilon \) for all \( n \geq 0 \). Let us show that any solution \( x(n, 0, x_0) \) of Eq. (3.21) satisfies the relation

\[ \lim_{n \to \infty} x(n, 0, x_0) = 0. \] (3.26)

For this take any vector \( x_0 \) and choose \( H \) such that

\[ \varphi_2(|x_0|) \leq \varphi_1(H). \]
The last inequality is valid due to (3.25). Therefore if relation (3.26) would be invalid then by virtue of (3.24) we have
\[ \sum_{j=1}^{N} \omega_{j}(|x_j|) \leq V(0,x_0). \] (3.27)
But the last inequality means that the series (3.27) must be divergent as \( N \to \infty \) which is a contradiction. Lemma 3.1 is proven. \( \square \)

3.4.1. Volterra convolution systems with dissipative nonlinearity
Consider system of nonlinear Volterra equations of convolution type
\[ x_{n+1} = -\sum_{j=0}^{n} A(n-j)g(x_j), \quad n \geq 0. \] (3.28)
In (3.28) vector \( x_n \in \mathbb{R}^r \), all matrices \( A(j) \) are symmetric, nonnegative definite, of dimension \( r \times r \), function \( g(x_j) = g(x^1_j, \ldots, x^r_j) \in \mathbb{R}^r \) and also
\[ g(0) = 0, \quad x'g(x) \geq 0, \quad x \in \mathbb{R}^r. \] (3.29)
For stability investigation consider the functional \( V(x_0, \ldots, x_n) \) equal
\[ V(x_0, \ldots, x_n) = 2x'_n g(x_n) - g'(x_n)A(-1)g(x_n) + \sum_{j=0}^{n} g'(x_j)A(n) \sum_{j=0}^{n} g(x_j) \]
\[ - \sum_{j=0}^{n} \left[ \sum_{l=j}^{n} g'(x_l) [A(n-j) - A(n-j-1)] \sum_{l=j}^{n} g(x_l) \right]. \] (3.30)
Here matrix \( A(-1) \) will be chosen later. Let us find first difference of the summands in the right hand side of the functional \( V \) along the solution of Eq. (3.28). We have
\[ 2x'_{n+1} g(x_{n+1}) - g'(x_{n+1})A(-1)g(x_{n+1}) \]
\[ - 2x'_n g(x_n) + g'(x_n)A(-1)g(x_n), \] (3.31)
further
\[ \sum_{j=0}^{n+1} g'(x_j)A(n+1) \sum_{j=0}^{n+1} g(x_j) - \sum_{j=0}^{n} g'(x_j)A(n) \sum_{j=0}^{n} g(x_j) \]
\[ = \sum_{j=0}^{n} g'(x_j)A(n+1) \sum_{j=0}^{n} g(x_j) + g'(x_{n+1})A(n+1)g(x_{n+1}) \]
\[ + 2 \sum_{j=0}^{n} g'(x_j)A(n+1)g(x_{n+1}) - \sum_{j=0}^{n} g'(x_j)A(n) \sum_{j=0}^{n} g(x_j) \]
\begin{align*}
&= \sum_{j=0}^{n} g'(x_j)[A(n+1) - A(n)] \sum_{j=0}^{n} g(x_j) \\
&\quad + 2g'(x_{n+1})A(n+1) \sum_{j=0}^{n} g(x_j) + g'(x_{n+1})A(n+1)g(x_{n+1}).
\end{align*}

The last summand in (3.30) is transformed as follows. Put

\begin{align*}
J_1(n) &= -\sum_{j=0}^{n} \sum_{l=j}^{n} g'(x_l)[A(n - j) - A(n - j - 1)] \sum_{l=j}^{n} g(x_l). 
(3.33)
\end{align*}

Then the first difference of the last summand in (3.33) will have the form $J_1(n + 1) - J_1(n)$. The functional $J_1(n + 1)$ can be written as

\begin{align*}
J_1(n + 1) &= -g'(x_{n+1})[A(0) - A(-1)]g(x_{n+1}) + J_2. 
(3.34)
\end{align*}

Here $J_2$ equal

\begin{align*}
J_2 &= -\sum_{j=0}^{n} \left[ \sum_{l=j}^{n+1} g'(x_l) \right] [A(n + 1 - j) - A(n - j)] \sum_{l=j}^{n+1} g(x_l) \\
&= -\sum_{j=0}^{n} \left[ g'(x_{n+1}) + \sum_{l=j}^{n} g'(x_l) \right] [A(n + 1 - j) - A(n - j)] \sum_{l=j}^{n} g(x_l) \\
&\quad \times \left[ g(x_{n+1}) + \sum_{l=j}^{n} g(x_l) \right]. 
(3.35)
\end{align*}

From (3.33)–(3.35) it follows that

\begin{align*}
J_1(n + 1) &= -g'(x_{n+1})[A(0) - A(-1)]g(x_{n+1}) + J_3 \\
&\quad - g'(x_{n+1}) \sum_{j=0}^{n} [A(n + 1 - j) - A(n - j)]g(x_{n+1}) \\
&\quad - 2g'(x_{n+1}) \sum_{j=0}^{n} [A(n + 1 - j) - A(n - j)] \sum_{l=j}^{n} g(x_l),
\end{align*}

where

\begin{align*}
J_3 &= -\sum_{j=0}^{n} \left[ \sum_{l=j}^{n} g'(x_l) \right] [A(n + 1 - j) - A(n - j)] \sum_{l=j}^{n} g(x_l). 
(3.36)
\end{align*}
Remark that by virtue of Eq. (3.28)
\[
\sum_{j=0}^{n} [A(n+1-j) - A(n-j)] \sum_{l=j}^{n} g(x_l)
\]
\[
= \sum_{l=0}^{n} \left[ \sum_{j=0}^{l} [A(n+1-j) - A(n-j)] \sum_{l=j}^{n} g(x_l) \right]
\]
\[
= \sum_{l=0}^{n} [A(n+1-l) - A(n-l)] g(x_l) = A(n+1) \sum_{l=0}^{n} g(x_l) + x_{n+1}. \quad (3.37)
\]

From (3.37) we have
\[
-2g'(x_{n+1}) \sum_{j=0}^{n} [A(n+1-j) - A(n-j)] \sum_{l=j}^{n} g(x_l)
\]
\[
-2g'(x_{n+1})A(n+1) \sum_{l=0}^{n} g(x_l) - 2g'(x_{n+1})x_{n+1}. \quad (3.38)
\]

Therefore the functional \( J_1(n+1) \) takes the following form:
\[
J_1(n+1) = -g'(x_{n+1})[A(n+1) - A(-1)]g(x_{n+1})
\]
\[
-2g'(x_{n+1})A(n+1) \sum_{l=0}^{n} g(x_l) - 2g'(x_{n+1})x_{n+1} + J_3. \quad (3.39)
\]

As a result relations (3.30)–(3.39) give the first difference for the function \( V \):
\[
V(n+1,x_0,\ldots,x_{n+1}) - V(n,x_0,\ldots,x_n) = -2x'_n g(x_n)
\]
\[
+ g'(x_n)A(-1)g(x_n) + \sum_{j=0}^{n} g'(x_j)[A(n+1) - A(n)] \sum_{j=0}^{n} g(x_j)
\]
\[
- \sum_{j=0}^{n} \left[ \left( \sum_{l=j}^{n} g'(x_l) \right) [A(n+1-j) - 2A(n-j) + A(n-j-1)] \sum_{l=j}^{n} g(x_l) \right]. \quad (3.40)
\]

From the form of functional (3.30) and its first difference (3.40) using Lemma 3.2 it follows.

**Theorem 3.4.** Let all the matrices \( A(-1),\ldots,A(n+1) \) are nonnegative definite, symmetric and also
\[
A(j+1) - A(j) \leq 0,
\]
\[
A(j+1) - 2A(j) + A(j-1) \geq 0, \quad j = 0,1,\ldots. \quad (3.41)
\]
The function \( g(x) \) satisfies conditions of (3.25) and also
\[
\omega(x) = 2x'g(x) - g'(x)A(-1)g(x) > 0, \quad x \neq 0,
\]
\[
\omega(x) \to \infty, \quad |x| \to \infty.
\]
Then the zero solution of Eq. (3.28) is asymptotically stable in the whole.

**Proof.** Under conditions (3.40) and (3.41) the functional in (3.30) is positive definite and its first difference is negative definite. Hence the zero solution of Eq. (3.28) is asymptotically stable. If condition (3.43) is valid then the solution \( x(n,0,x_0) \) be bounded for any initial condition \( x_0 \) and all \( n \geq 0 \). Hence by virtue of negative definiteness of the first difference of the functional \( V \) the solution \( x(n,0,x_0) \to 0 \) as \( n \to \infty \). Theorem 3.4 is proven.

**Example 3.3.** Consider a two dimensional nonlinear systems of Volterra difference equations:
\[
x_1(n+1) = -b \sum_{i=0}^{n} x_1(i) \left( -\frac{1}{2} \sin^2 x_2(i) + 1 \right),
\]
\[
x_2(n+1) = -b \sum_{i=0}^{n} x_2(i) \left( -\frac{1}{2} \sin^2 x_1(i) + 1 \right), \quad n \geq 0.
\]
On the strength of Theorem 3.4 the zero solution of (3.44) is asymptotically stable in the whole for \( 0 \leq b < 2 \).

3.4.2. Nonconvolution systems with dissipative nonlinearity

Let us obtain stability conditions in the whole of the zero solution for Volterra difference equations
\[
x_{n+1} = -\sum_{j=0}^{n} A(n,j)g(x_j), \quad n \geq j \geq 0, \quad x_n \in \mathbb{R}.
\]
In Eq. (3.45) all matrices \( A(n,j) \) are symmetric and nonnegative definite. Function \( g(x) \) satisfies conditions (3.29). Let us continue the matrices \( A(n,j) \) for all values of arguments \( (n-1,-1), (n-1,n) \) such that as before they will be symmetric and nonnegative definite.

Further introduce the functional
\[
V(n,x_0,\ldots,x_n) = 2x'_n g(x_n) - g'(x_n)A(-1,n)g(x_n)
\]
\[
+ \sum_{j=0}^{n} g'(x_j)[A(n-1,-1)] - \sum_{j=0}^{n} g(x_j) - \sum_{j=0}^{n} \left[ \sum_{l=j}^{n} g'(x_l) \right]
\]
\[
\times [A(n-1,j-1) - A(n-1,j)] \sum_{l=j}^{n} g(x_l).
\]
\[
(3.46)
\]
It is clear that if the matrices $A(n,j)$ in Eq. (3.46) do depend only on differences of the arguments $n - j$, then the functional (3.46) coincides with those of (3.30). Let us calculate first differences of summands in the right hand side of (3.46) along the solutions of Eq. (3.45). We have
\[2x'_{n+1}g(x_{n+1}) - g'(x_{n+1})A(n,n+1)g(x_{n+1}) - 2x'_ng(x_n) + g'(x_n)A(n-1,n)g(x_n).\]

Further similar to (3.32):
\[
\sum_{j=0}^{n+1} g'(x_j)A(n,-1)\sum_{j=0}^{n+1} g(x_j) - \sum_{j=0}^n g'(x_j)[A(n-1,-1)]\sum_{j=0}^n g(x_j)
\]
\[=\sum_{j=0}^n g'(x_j)[A(n,-1) - A(n+1,-1)]\sum_{j=0}^n g(x_j)
+ 2\sum_{j=0}^n g'(x_j)A(n-1)g(x_{n+1}) - \sum_{j=0}^n g'(x_j)A(n,1)\sum_{j=0}^n g(x_j)
\]
\[=\sum_{j=0}^n g'(x_j)[A(n,-1) - A(n+1,-1)]\sum_{j=0}^n g(x_j)
+ 2g'(x_{n+1})A(n-1)\sum_{j=0}^n g(x_j) + g'(x_{n+1})A(n-1)g(x_{n+1}).\]

In order to transform last summand in (3.46) put
\[J_1(n) = -\sum_{j=0}^n \left[ \left( \sum_{l=j}^{n+1} g'(x_l) \right) [A(n-1,j-1) - A(n-1,j)] \sum_{l=j}^n g(x_l) \right].\]

Then first difference of the last summand equals $J_1(n+1) - J_1(n)$. For the functional $J_1(n+1)$ we have
\[J_1(n+1) = -g'(x_{n+1})[A(n,n) - A(n,n+1)]g(x_{n+1}) + J_2, \tag{3.47}\]
here
\[J_2 = -\sum_{j=0}^{n+1} \left[ \left( \sum_{l=j}^{n+1} g'(x_l) \right) [A(n,j-1) - A(n,j)] \sum_{l=j}^{n+1} g(x_l) \right]
\[= -\sum_{j=0}^n \left( g'(x_{n+1}) + \sum_{l=j}^n g'(x_l) \right) w[A(n,j-1) - A(n,j)]
\times g(x_{n+1}) + \sum_{l=j}^n g(x_l) \right]. \tag{3.48}\]
From (3.47) and (3.48) it follows that

\[ J_1(n+1) = -g'(x_{n+1})[A(n,n) - A(n,n+1)]g(x_{n+1}) + J_3 \]

\[ - g'(x_{n+1}) \sum_{j=0}^{n} [A(n,j - 1) - A(n,j)]g(x_{n+1}) \]

\[ - 2g'(x_{n+1}) \sum_{j=0}^{n} [A(n,j - 1) - A(n,j)] \sum_{l=j}^{n} g(x_l), \]

here

\[ J_3 = -\sum_{j=0}^{n} \left[ \left( \sum_{l=j}^{n} g'(x_l) \right) [A(n,j - 1) - A(n,j)] \sum_{l=j}^{n} g(x_l) \right]. \]  (3.49)

Further by virtue of Eq. (3.45):

\[ \sum_{j=0}^{n} [A(n,j - 1) - A(n,j)] \sum_{l=j}^{n} g(x_l) = \sum_{l=0}^{n} \sum_{j=0}^{l} [A(n,j - 1) - A(n,j)]g(x_l) \]

\[ = \sum_{l=0}^{n} [A(n,l - 1) - A(n,l)]g(x_l) = A(n,-1) \sum_{l=0}^{n} g(x_l) + x_{n+1}, \]

from here it follows:

\[ -2g'(x_{n+1}) \sum_{j=0}^{n} [A(n,j - 1) - A(n,j)] \sum_{l=j}^{n} g(x_l) \]

\[ = -2g'(x_{n+1})A(n,-1) \sum_{l=0}^{n} g(x_l) - 2g'(x_{n+1})x_{n+1}. \]  (3.50)

As a result the functional \( J_1(n+1) \) takes the form

\[ J_1(n+1) = -g'(x_{n+1})[A(n,n) - A(n,n+1)]g(x_{n+1}) \]

\[ - 2g'(x_{n+1})A(n,-1) \sum_{l=0}^{n} g(x_l) - 2x_{n+1}'g(x_{n+1}) + J_3. \]  (3.51)

Relations from (3.46) to (3.51) give the following expression for the first difference of the functional \( V \) along the trajectories of Eq. (3.45):

\[ V(n+1,x_0,\ldots,x_{n+1}) - V(n,x_0,\ldots,x_n) \]

\[ = -2x_n'g(x_n) + g'(x_n)A(n-1,n)g(x_n) \]
\[ + n \sum_{j=0}^{n} g'(x_j)\{A(n, -1) - A(n - 1, -1)\} \sum_{j=0}^{n} g(x_j) \]
\[ - \sum_{i=0}^{n} \left[ \sum_{j=i}^{n} g'(x_j)\{A(n, i - 1) - A(n, i) - A(n - 1, i - 1)\} \sum_{j=i}^{n} g(x_j) \right]. \]

Now designate by \( \omega_1(x) \) the function:
\[
\omega_1(x) = 2x'g(x) + g'(x_n)A(n - 1, n)g(x).
\]

Considerations of this section lead us to the following theorem.

**Theorem 3.5.** Let the matrices:
\[
A(n, j) \quad \text{for} \quad n \geq j \geq 0 \quad \text{and} \quad A(n - 1, n), A(n - 1, -1) \quad \text{for} \quad n \geq 0,
\]
be symmetric, nonnegative definite and also the following conditions are valid:
\[
A(n, -1) - A(n - 1, -1) \leq 0, \quad n \geq 0,
\]
\[
A(n - 1, i - 1) - A(n - 1, i) \leq 0, \quad n \geq i \geq 0,
\]
\[
A(n, i - 1) - A(n, i) - A(n - 1, i - 1) + A(n - 1, i) \geq 0, \quad n \geq i \geq 0,
\]
\[
\omega_1(x) > 0, \quad x \neq 0; \quad \omega_1(x) \to \infty, \quad |x| \to \infty.
\]

Then the zero solution of Eq. (3.45) will be asymptotically stable in the whole.

### 3.5. Exponential stability of discrete Volterra systems

Let us consider a discrete homogeneous linear Volterra equation:
\[
x_{n+1} = \sum_{j=n_0}^{n} A_{n, j}x_j, \quad n \geq n_0, \quad x_{n_0} = x_0,
\]
where \( n_0 \in N_0, x_n \in \mathbb{R}^r \) and \( A_{n, j} \in \mathbb{R}^{r \times r} \). Denote by \( x(n, n_0, x_0) \) the solution of the problem (3.53). Let us give the following definition.

**Definition 3.2.** The solution \( x(n, n_0, x_0) \) of Eq. (3.53) is called exponentially stable with respect to the perturbations of the initial value \( x_0 \) if there exist constants \( C > 0 \) and \( \gamma \in (0, 1) \) such that
\[
|x(n, n_0, x_0)| \leq C|x_0|^\gamma, \quad \forall x_0 \in \mathbb{R}^r,
\]
where \( |\cdot| \) is a vector norm in the space \( \mathbb{R}^r \).

Due to the linearity of Eq. (3.53), if one solution is exponentially stable the same is true for all the solutions and (3.53) is said to be exponentially stable.

It can be easily seen that the exponential stability implies the uniform asymptotic stability [15].
Moreover, for finite order linear difference equations also the converse is true, that is the exponential stability is equivalent to the uniform asymptotic stability and so, in this case, the exponential stability is completely characterized.

Unfortunately, for the Volterra difference equations the latter is no true and until now several attempts to derive conditions of exponential stability have been made. Such attempts lead us to some results which regard only linear equations of convolution type and with summable kernels. We prove necessary and sufficient conditions for the exponential stability.

This allows us to obtain sufficient conditions directly in terms of the coefficients of the given equation, whose effectiveness is tested by applying them to equations with exponentially decreasing coefficients.

We wish to stress that, to the best of our knowledge, the mentioned results on the exponential stability are the first ones that hold also for equations of nonconvolution type.

3.5.1. Necessary and sufficient conditions of exponential stability

In this section necessary and sufficient conditions for the exponential stability are derived. To this purpose, let us introduce $Z_{n,j}$, the resolvent matrix of Eq. (3.53), that is defined by the relations [46]

$$
Z_{n+1,j} = \sum_{l=j}^{n} A_{n,l} Z_{l,j}, \quad n \geq j,
$$

$$
Z_{j,j} = I,
$$

where $I$ is the identity matrix of order $r \times r$. Then it results:

$$
x(n,n_0,x_0) = Z_{n,n_0} x_0.
$$

Therefore Eq. (3.53) is exponentially stable if and only if:

$$
|Z_{n,j}| \leq \gamma^{n-j}, \quad n \geq j, \quad 0 < \gamma < 1;
$$

where $\lambda$ is a positive constant and $|Z_{n,j}|$ is a norm in the space of matrices induced by the vector norm $|\cdot|$ in the space $\mathbb{R}^r$.

Firstly, in order to underline a crucial difference between the discrete Volterra equations and the finite order difference equations, we want to give the following result. Let us observe that, for any fixed $m$ such that $j \leq m \leq n$, Eq. (3.55) can be written as

$$
Z_{n+1,j} = \sum_{l=m}^{n} A_{n,l} Z_{l,j} + \sum_{l=j}^{m-1} A_{n,l} Z_{l,j}.
$$

Then each column of $Z_{n+1,j}$ satisfies a Volterra discrete equation of the type $Z_{n+1,j} = \sum_{j=m}^{n} A_{n,j} Z_{j} + p_n$ and by using the expression of its solution given in [15] we can prove that the resolvent matrix satisfies the following identity:

$$
Z_{n,j} = Z_{n,j} + \sum_{k=l}^{n-1} Z_{n,k+1} \sum_{h=j}^{l-1} A_{k,h} Z_{h,j}, \quad n \geq l \geq j.
$$
Then, the resolvent matrix satisfies the group property

\[ Z_{n,j} = Z_{n,l} Z_{l,j}, \]

if and only if Eq. (3.53) reduces to a finite order equation. This property is very important in the theory of finite order difference equations because it allows to prove that uniform asymptotic stability implies exponential stability. Unfortunately, because of (3.57), this is not true in the case of Volterra discrete equations.

In order to derive conditions of the exponential stability for Volterra discrete equations, let us introduce two Banach spaces \( L^\gamma \) and \( C^\gamma \), for \( 0 < \gamma < 1 \), defined by

\[
L^\gamma = \left\{ \{ f_n \}_{n \in \mathbb{N}} : f_n \in \mathbb{R}^r, \sum_{n=0}^{\infty} |f_n|^{\gamma^{-n}} < \infty \right\},
\]

\[
C^\gamma = \left\{ \{ f_n \}_{n \in \mathbb{N}} : f_n \in \mathbb{R}^r, \sup_{n \geq 0} \{|f_n|^{\gamma^{-n}}\} < \infty \right\}.
\]

Norms in the spaces \( L^\gamma \) and \( C^\gamma \) are given by the relations

\[
\| f \|_{L^\gamma} = \sum_{n=0}^{\infty} |f_n|^{\gamma^{-n}}, \quad (3.58)
\]

\[
\| f \|_{C^\gamma} = \sup_{n \geq 0} \{|f_n|^{\gamma^{-n}}\}. \quad (3.59)
\]

Let us introduce also the operator \( F \) in \( L^\gamma \):

\[ F : f \in L^\gamma \to y = \{ y_n \}, \]

where

\[ y_0 = 0, \]

\[ y_n = \sum_{j=0}^{n-1} Z_{n,j+1} f_j, \quad n \geq 1, \quad y_n \in \mathbb{R}^r. \quad (3.60) \]

Now, we can prove the following lemma.

**Lemma 3.2.** Assume that the operator \( F(f) \) maps \( L^\gamma \) into \( C^\gamma \) for some \( \gamma \in (0, 1) \). Then the operator \( F \) is bounded.

**Proof.** Let us consider the sequence \( \{ F_n(f) \}_{n \in \mathbb{N}} \) of operators mapping \( L^\gamma \) into \( \mathbb{R}^r \), defined by

\[
\{ F_n(f) \}_{n \in \mathbb{N}} = \gamma^{-n} \sum_{j=0}^{n-1} Z_{n,j+1} f_j.
\]

For any fixed point \( \{ f \} \in L^\gamma \), due to the assumption of Lemma 3.2, the sequence \( \{ F_n(f) \}_{n \in \mathbb{N}} \) is uniformly bounded with respect to \( n \), i.e.:

\[
\sup_{n \geq 0} \{ |F_n(f)| \} < \infty.
\]
Hence, by using the Banach–Steinhaus theorem [24], we can conclude that the norms \( \{\|F_n\|\}_{n \in N} \) of the operators \( \{F_n(f)\}_{n \in N} \) are uniformly bounded with respect to \( n \), i.e. there exists a constant \( K > 0 \) such that:

\[
\sup_{n \geq 0} \{\|F_n(f)\|\} \leq K.
\]

Therefore, taking into account (3.60), it follows:

\[
\sup_{n \geq 0} \{\gamma^{-n}|y_n|\} = \|y\|_{C^\gamma} \leq K \|f\|_{L^\gamma}.
\]

Lemma 3.2 is proven. \( \square \)

**Theorem 3.6.** Eq. (3.53) is exponentially stable if and only if the operator \( F \) maps \( L^\gamma \) into \( C^\gamma \) for some \( \gamma \in (0, 1) \).

**Proof.** Let us assume that there exists \( \gamma \in (0, 1) \) such that \( F \) maps \( L^\gamma \) into \( C^\gamma \). Then, by virtue of Lemma 3.2, the operator \( F \) is bounded.

From here, taking into account (3.60) it follows that

\[
\sup_{n \geq 0} \{\gamma^{-n}|y_n|\} = \sup_{n \geq 0} \left\{ \gamma^{-n} \left| \sum_{j=0}^{n-1} Z_{n,j+1} f_j \right| \right\} \leq K \sum_{j=0}^\infty \gamma^{-j} |f_j|, \quad \forall \{f\} \in L^\gamma.
\]

Let us choose the following sequence \( \{f_j\}_{j \in N} \) belonging to \( L^\gamma \):

\[
f_j = \begin{cases} 1 & \text{for } j = l, \\ 0 & \text{for } j \neq l,
\end{cases}
\]

where \( l \) is such that \( l \leq n - 1 \). Then, from the previous inequality we obtain the estimate:

\[
\gamma^{-n}|Z_{n,l+1}| \leq K \gamma^{-l}.
\]

Hence the resolvent matrix \( Z_{n,l} \) satisfies estimate (3.56) and so Eq. (3.53) is exponentially stable.

By converse, let us assume that the equation is exponentially stable, so that (3.56) holds. Then, by virtue of (3.60) we can conclude that

\[
\gamma^{-n}|y_n| = \gamma^{-n} \left| \sum_{j=0}^{n-1} Z_{n,j+1} f_j \right| \leq K \gamma^{-n} \sum_{j=0}^{n-1} \gamma^{n-j-1} |f_j| = K \gamma^{-1} \sum_{j=0}^{n-1} \gamma^{-j} |f_j|.
\]

Therefore

\[
\sup_{n \geq 0} \{\gamma^{-n}|y_n|\} \leq K \gamma^{-1} \sup_{n \geq 0} \left\{ \sum_{j=0}^{n-1} \gamma^{-j} |f_j| \right\} = K \gamma^{-1} \|f\|_{L^\gamma} < \infty,
\]

and the theorem is proven. \( \square \)
At present, we can conclude that exponential stability is a very important property to control how the solution is influenced by perturbations on the equation, and hence to analyse the stability of methods for Volterra equations.

3.6. The Lyapunov function method

Let us consider the Volterra equation

$$x_{n+1} = \sum_{j=0}^{n} F(n,j,x_j), \quad x_j \in \mathbb{R}^r, \quad n \geq 0, \quad F(n,j,0) = 0,$$

(3.61)

where the function $F(n,j,x)$ is continuous in $x$ and bounded in $n$ and $j$, and the initial condition $x_0 \in \mathbb{R}^r$. We assume that the solution $x_n = x(n,0,x_0)$ of equation (3.61) exists for all $n \geq 0$ and continuously depends on $x_0$.

The zero solution of system (3.61) is said to be stable if there exists a $p_{SO} > 0$ for any $p_{SI} > 0$ such that the condition $|x_0| < p_{SO}$ implies the relation $|x(n,0,x_0)| < p_{SI}$, $n \geq 0$, and to be asymptotically stable in the whole if it is stable and

$$\lim_{n \to \infty} x(n,0,x_0) = 0, \quad x_0 \in \mathbb{R}^r.$$

(3.62)

Theorem 3.7. Let there exist a continuous function $V(n,x)$ of $x$ satisfying the conditions

$$V(n,x_n) \geq W_1(|x_n|), \quad V(n,0) = 0,$$

(3.63)

$$v(n + 1) - v(n) \leq - q(n)W(v(n + 1)) + \sum_{j=0}^{n} \gamma(n,j)W(v(j)),$$

(3.64)

where $v(n) = V(n,x)$, $\gamma(n,j) \geq 0$, $\inf_{n \geq 0} \{q(n)\} > 0$, and the functions $W$ and $W_1$ are continuous, $W(0) = W_1(0) = 0$, and monotonically increase. Furthermore, let

$$\lim_{n \to \infty} q^{-1}(n) \sum_{j=0}^{n} \gamma(n,j) = 0.$$

(3.65)

Then the zero solution of system (3.61) is stable.

Proof. Choose a fixed constant $0 < z < 1$. By (3.65), there exists an instant $n_0$ such that

$$xzq(n) - \sum_{j=0}^{n} \gamma(n,j) \geq 0, \quad \forall n \geq n_0 \geq 0.$$

(3.66)

Let $\varepsilon > 0$. By the conditions imposed on $V(n,x)$, there exists a $\delta_1 > 0$ such that $V(n,x) < W_1(\varepsilon)$ for $n \leq n_0$ and $|x| \leq \delta_1$. Now choose a $\delta$, $0 < \delta \leq \delta_1$, such that $|x(n,0,x_0)| < \delta_1$ for $|x_0| \leq \delta$ and for all $n$, $0 \leq n \leq n_0$. Such a $\delta$ exists, because $x(n,0,x_0)$ continuously depends on $x_0$. Therefore, $v(n) = V(n,x(n,0,x_0)) < W_1(\varepsilon)$ for
We now show that \( v(n) < W_1(\varepsilon) \) for all \( n \geq 0 \). Assuming the contrary, i.e., there exists a first instant \( n_1 > n_0 \) such that
\[
v(n) < v(n_1), \quad n < n_1, \quad v(n_1) \geq W_1(\varepsilon),
\]
by virtue of (3.64) and (3.65), we find that
\[
v(n_1) - v(n_1 - 1) \leq -q(n_1 - 1)W_1(v(n_1)) + \sum_{j=0}^{n_1-1} \gamma(n_1 - 1, j)W(v(j))
\]
\[
\leq - \left[ q(n_1 - 1) - \sum_{j=0}^{n_1-1} \gamma(n_1 - 1, j) \right] W_1(v(n_1))
\]
\[
\leq - \left[ 1 - q^{-1}(n_1 - 1) \sum_{j=0}^{n_1-1} \gamma(n_1 - 1, j) \right] q(n_1 - 1)W_1(v(n_1))
\]
\[
\leq -(1-z)q(n_1 - 1)W_1(v(n_1)) < 0 \Rightarrow v(n_1) < v(n_1 - 1).
\]
This contradiction shows that \( v(n) < W_1(\varepsilon) \) for all \( n \geq 0 \). Hence, by virtue of (3.63), \( W_1(|x_n|) \leq V(n, x_n) \leq W_1(\varepsilon) \). Therefore, since the function \( W_1 \) is monotonic, \( |x_n| \leq \varepsilon \) for all \( n \geq 0 \).

This completes the proof of Theorem 3.7. \( \Box \)

**Theorem 3.8.** Along with the conditions of Theorem 3.7, if the function \( W_1 \) satisfies the condition
\[
W_1(|x_n|) \to \infty \quad \text{as} \quad |x_n| \to \infty,
\]
then the zero solution of system (3.61) is asymptotically stable on the whole.

**Proof.** Taking an arbitrary \( x_0 \in \mathbb{R}^r \), let us show that the solution \( x(n, 0, x_0) \) satisfies relation (3.62). For this, it suffices to show that there exists an instant \( n(\varepsilon) \) for any \( \varepsilon > 0 \) such that
\[
|x(n, 0, x_0)| < \varepsilon, \quad \forall n \geq n(\varepsilon).
\]
Let us consider the solution \( x(n, 0, x_0) \) on the interval \([0, n_0]\), where \( n_0 \) is defined by relation (3.66). Let us find a number \( \beta \) (it exists by condition (3.67)) such that:
\[
W_1(\beta) \geq \max_{0 \leq n \leq n_0} \{ V(n, x(n, 0, x_0)) \}.
\]
Reasoning as in the proof of Theorem 3.7 (and replacing \( W_1(\varepsilon) \) by \( W_1(\beta) \)), we find that:
\[
v(n) = V(n, x(n, 0, x_0)) < W_1(\beta).
\]
Let us take an $n_2 > n_0$ such that:

$$q^{-1}(n) \sum_{j=0}^{n} \gamma(n, j) < W(W_1(\varepsilon))/2W(W_1(\beta)) \quad \text{for all } n \geq n_2. \quad (3.70)$$

We now show that there is no point $\tau > n_2$ at which $v(\tau) > W_1(\varepsilon)$. Assume that such a point $\tau > n_2$ exists. Then $W(v(\tau)) > W(W_1(\varepsilon))$. Therefore, by virtue of (3.64) and (3.70), we have

$$v(\tau) - v(\tau - 1) \leq -q(\tau - 1)W(v(\tau)) + \sum_{j=0}^{\tau-1} \gamma(\tau - 1, j)W(v(j))$$

$$\leq -q(\tau - 1)W(v(\tau)) + W(W_1(\beta))\sum_{j=0}^{\tau-1} \gamma(\tau - 1, j)$$

$$\leq q(\tau - 1)W(v(\tau)) + W(W_1(\beta))m(\tau - 1)W(W_1(\varepsilon))/2W(W_1(\beta))$$

$$\leq -q(\tau - 1)[W(v(\tau)) - W(W_1(\varepsilon))]/2$$

$$\leq -q(\tau - 1)W_1(\varepsilon))/2 < 0, \quad \tau > n_2. \quad (3.71)$$

This inequality implies that $v(n) \leq W_1(\varepsilon)$ for all $n \geq \tau - 1$ if $v(\tau - 1) \leq W_1(\varepsilon)$ at the point $(\tau - 1)$.

What now remains is to show that there exists a point $\tau_1 \geq n_2$ at which $v(\tau_1) \leq W_1(\varepsilon)$. Let us take an instant $n_3$ such that

$$\sum_{j=n_2}^{n_3} q(j) > 4W_1(\beta)/W(W_1(\varepsilon)). \quad (3.72)$$

If a point $\tau_1 > n_2$ at which $v(\tau_1) \leq W_1(\varepsilon)$ does not exist, then $v(\tau) > W_1(\varepsilon)$ for all $\tau > n_2$, i.e.

$$v(\tau) - v(\tau - 1) \leq -q(\tau - 1)W(W_1(\varepsilon))/2, \quad \tau > n_2,$$

by (3.71). Summing this inequality from $\tau = n_2 + 1$ to $\tau = n_3 + 1$, by virtue of (3.72), we obtain

$$v(n_3 + 1) - v(n_2) \leq (-W(W_1(\varepsilon))/2\sum_{\tau=n_2}^{n_3} m(\tau) \leq -2W_1(\beta). \quad (3.73)$$

But, by virtue of (3.69) and (3.73):

$$v(n_3 + 1) \leq v(n_2) - 2W_1(\beta) \leq -W_1(\beta) < 0.$$

By estimate (3.62), the last inequality is impossible. This completes the proof of Theorem 3.8. \hfill \Box

**Example 3.4.** Let us consider the scalar Volterra equation

$$x_{n+1} = x_n x_n + \sum_{j=0}^{n} a_{n,j}x_j, \quad n \geq 0. \quad (3.74)$$
Let $V(n, x(n, 0, x_0)) = |x_n|$. Then
\[ \Delta V = |x_{n+1}| - |x_n|. \] (3.75)

By virtue of (3.74), we have
\[ |x_n| \geq |x_n|^{-1} \left[ |x_{n+1}| - \sum_{j=0}^{n} |a_{n,j}x_j| \right], \quad n \geq 0. \] (3.76)

Therefore, by virtue of (3.75) and (3.76):
\[ \Delta V \leq |x_{n+1}|(1 - |x_n|^{-1}) + |x_n|^{-1} \sum_{j=0}^{n} |a_{n,j}x_j|. \]

Hence, by the conditions of Theorems 3.7 and 3.8, the system is asymptotically stable on the whole if
\[ \sup_{n \geq 0} \{ |x_n| \} < 1 \quad \text{and} \quad \lim_{n \to \infty} \sum_{j=0}^{n} |a_{n,j}| = 0. \]

3.7. The comparison principle

Let us consider the linear Volterra equation:
\[ x_{n+1} = A_n x_n + \sum_{j=0}^{n} R_{n,j} x_j, \quad n \geq 0, \quad x_n \in \mathbb{R}^r. \] (3.77)

where $R_{n,j}$ and $A_n$ are given sequences of $r \times r$ matrices. Along with equation (3.77), let us consider the inequality:
\[ z_{n+1} \geq A_n z_n + \sum_{j=0}^{n} |R_{n,j}| z_j, \quad n \geq 0, \quad z_n \in \mathbb{R}^r. \] (3.78)

Inequality (3.78) is interpreted element by element (i.e., if $u \geq v, u \in \mathbb{R}^r$, and $v \in \mathbb{R}^r$, then $u_j \geq v_j$ for $j = 1, \ldots, r$) and the matrix $|R_{n,j}|$ is formed from the moduli of the corresponding elements of the matrix $R_{n,j}$. Assume that all elements of the matrix $A_n$ are nonnegative. Let $x_0$ and $z_0$ denote the initial conditions of relations (3.77) and (3.78) for $n = 0$.

Assume that $z(0) \geq |x_0|$ and there exists a positive solution $z_n > 0, n \geq 0$, of inequality (3.78). Then
\[ |x_n| \leq z_n, \quad n \geq 0, \] (3.79)

where the vector $|x_n|$ is the modulus of the vector $x_n$.

To demonstrate (3.79), taking a number $0 < \varepsilon < 1$, let us study the equation for $x_n^\varepsilon$:
\[ x_{n+1}^\varepsilon = (A_n - \varepsilon I)x_n^\varepsilon + \sum_{j=0}^{n} R_{n,j} x_j^\varepsilon, \quad n \geq 0, \quad x_n^\varepsilon = (1 - \varepsilon)x_0, \]

where $I$ is an $r \times r$ unit matrix.
Obviously, $z_0 > |x_0^i|$. Let us assume that a first instant $n_0 + 1$ exists such that

$$|x_{n_0+1}^i| > z_{n_0+1}^i,$$

(3.80)

for some coordinate $x_{n_0+1}^i$. Moreover

$$|x_n^i| \leq z_n^i, \quad n \leq n_0, \quad j = 1, \ldots, r.$$  

(3.81)

Without loss of generality, we can assume that

$$x_{n_0+1}^i > 0.$$  

(3.82)

Hence, by virtue of (3.77), (3.78) and (3.80)–(3.82):

$$0 < x_{n_0+1}^i - z_{n_0+1}^i \leq \left[ (A_{n_0} - \varepsilon I)x_{n_0}^i + \sum_{j=0}^{n_0} (R_{n,j}x_j^i - |R_{n,j}| z_j) \right]^i \leq - \varepsilon x_{n_0}^i,$$

where $[ \cdot ]^i$ denotes the $i$th component of the expression within square brackets. This contradiction shows that

$$|x_n^i| \leq z_n, \quad \forall n \geq 0.$$

Now taking the limit as $\varepsilon \to \infty$, we obtain (3.79).

**Corollary 1.** If the conditions

$$0 \leq A_{n}^{ii} \leq 1 - \sum_{j,j \neq i} A_{n}^{ij} - \sum_{j,k} |R_{n,j}^k|,$$

(3.83)

hold for all $i$, then system (3.77) is stable.

Indeed, if (3.83) holds, then the vector $z_n$ with identical components $C > 0$ satisfies (3.78).

**Remark 3.5.** For system (3.77) to be stable, it is necessary and sufficient that the solution $x(n,0,x_0)$ be bounded under any initial condition $x_0$, $|x_0| < \infty$.

To prove the sufficiency part, let us introduce an operator $U_n$ by the formula

$$x(n,0,x_0) = U_n x_0.$$  

Obviously, the operator $U_n$ is continuous (because $x(n,0,x_0)$ continuously depends on the initial conditions $x_0$) and $\sup_{n \geq 0} \{|U_n x_0|\} < \infty$, for all $x_0$, $|x_0| < \infty$.

Hence, the norms $\|U_n\|$ of the operators $U_n$, by the Banach–Steinhaus theorem, are uniformly bounded. Therefore

$$|x(n,0,x_0)| \leq \|U_n\| |x_0|,$$

implying the stability of system (3.77). Conversely, if system (3.77) is stable, then every solution (with $|x_0| < \infty$) is bounded.
If the Volterra system (3.61) is represented as
\[ x_{n+1} = f(x_n) + g(n,x_0,\ldots,x_n), \quad f(0) = 0, \quad g(n,0,\ldots,0) = 0, \]  
then the function \( V \) can be taken to be the Lyapunov function for the one-step process
\[ y_{n+1} = f(y_n), \quad n \geq 0. \]  
(3.85)
For example, let the function \( V(y_n) = |y_n| \) satisfy the condition
\[ V(y_{n+1}) - V(y_n) \leq -|y_{n+1}|, \quad n \geq 0. \]
Considering the solution of Eq. (3.85) for \( n \geq m \) under the initial condition \( y_m = x_m \), we find that:
\[ V(x_{m+1}) - V(x_m) = V(x_{m+1}) - V(x_m) \pm |y_{m+1}| \]
\[ = |x_{m+1}| - |y_{m+1}| + |y_{m+1}| - |y_m| \]
\[ \leq -|f(y_m)| + |x_{m+1}| - |y_{m+1}| \pm |x_{m+1}| \]
\[ = -|x_{m+1}| + (|x_{m+1}| - |f(x_m)|) \]
\[ + (|x_{m+1}| - |y_{m+1}|) \leq - |x_{m+1}| + 2|g(m,x_0,\ldots,x_m)|. \]
Thus, if
\[ |g(n,x_0,\ldots,x_n)| \leq \sum_{j=0}^{n} \gamma(n,j)|x_j|, \]
and the function \( \gamma(n,j) \) satisfies condition (3.65), then the zero solution of system (3.84) is asymptotically stable on the whole.

3.8. The functional method

Let us consider the Volterra equation:
\[ x_{n+1} = f_n + \sum_{j=0}^{n} K(n,j)g(x_j), \quad n \geq 0, \]  
(3.86)
where \( x_n \in \mathbb{R}^r \), the sequence \( K(n,j) \) of \( r \times r \) matrices and perturbations \( f_n \in \mathbb{R}^r \) is given, and
\[ \sum_{n=0}^{\infty} |f_n| < \infty, \]
(3.87)
\[ \sum_{l=0}^{\infty} \|K(l+n+1,n+1)\| \leq C_1 < 1. \]  
(3.88)
Here \( \|K\| \) is the matrix norm induced by the vector norm in \( \mathbb{R}^r \) and \( C_1 > 0 \) is a constant.
The function $g(x)$ satisfies the estimate
\[ |g(x)| \leq |x|. \] (3.89)

We now prove that the solution of Eq. (3.86) for any initial condition $x_0$, $|x_0| < \infty$, under assumptions (3.87)–(3.89) tends to zero as $n \to \infty$. Let us introduce the functional $V = V(n, x_0, \ldots, x_n)$:
\[ V = C \sum_{j=0}^{n} \sum_{l=n-j}^{\infty} |K(l + j)g(x_j)|. \] (3.90)

Here the constant $C$ is such that
\[ C > 1, \quad CC_1 < 1. \] (3.91)

Let us compute the increment of $V$ along the trajectory of system (3.86). We have
\[
\Delta V = C \sum_{j=0}^{n+1} \sum_{l=n+1-j}^{\infty} |K(l + j)g(x_j)| - C \sum_{j=0}^{n} \sum_{l=n-j}^{\infty} |K(l + j)g(x_j)|
\]
\[
= C|g(x_{n+1})| \sum_{l=0}^{\infty} \|K(l + n + 1, n + 1)\|
\]
\[
+ C \sum_{j=0}^{n} \left[ \left( \sum_{l=n+1-j}^{\infty} - \sum_{l=n-j}^{\infty} \right) |K(l + j)g(x_j)| \right]
\]
\[
= C|g(x_{n+1})| \sum_{l=0}^{\infty} \|K(l + n + 1, n + 1)\| - C \sum_{j=0}^{n} |K(n, j)g(x_j)|. \] (3.92)

By virtue of (3.86):
\[
- \sum_{j=0}^{n} |K(n, j)g(x_j)| \leq |f_n| - |x_{n+1}| \leq |f_n| - |g(x_{n+1})|. \] (3.93)

From (3.89), (3.92) and (3.93) we obtain
\[
\Delta V \leq CC_1|g(x_{n+1})| + C|f_n| - C|x_{n+1}| \leq |x_{n+1}|(CC_1 - C) + C|f_n|. \]

Summing both sides of the last inequality from zero to $n$, we obtain
\[
V(n + 1) - V(0) \leq (CC_1 - C) \sum_{j=0}^{n} |x_{j+1}| + C \sum_{j=0}^{n} |f_j|. \] (3.94)

Since $V \geq 0$ by (3.90), from (3.94) we find that
\[
\sum_{j=0}^{\infty} |x_j| < \infty,
\]
i.e., $x_j \to 0$ as $j \to \infty$. 

Interestingly
\[
\sum_{j=0}^{\infty} \left[ (1 - CC_1) |g(x_{j+1})| + (C - 1) \sum_{l=0}^{j} |K(j, l)g(x_l)| \right] \leq V(0, x_0) + \sum_{j=0}^{n} |f_j|.
\]

This estimates follows from the relations:
\[
\Delta V \leq CC_1 |g(x_{n+1})| - C \sum_{j=0}^{n} |K(n, j)g(x_j)| \pm \sum_{j=0}^{n} |K(n, j)g(x_j)|
\]
\[
\leq CC_1 |g(x_{n+1})| - (C - 1) \sum_{j=0}^{n} |K(n, j)g(x_j)| + |f_n| - |g(x_{n+1})|
\]
\[
\leq |f_n| - (1 - CC_1) |g(x_{n+1})| - (C - 1) \sum_{j=0}^{n} |K(n, j)g(x_j)|.
\]

Here “±” means “adding and subtracting”.

3.8.1. The asymptotic behavior of linear systems with variable parameters and aftereffect

For equations with constant duration finite aftereffect, the second Lyapunov method allows one to state stability conditions in terms of the existence of Lyapunov functions depending on finitely many arguments.

The construction of a formal procedure providing explicit expressions for Lyapunov functionals and hence for stability conditions for the above finite-difference equation can be found in [34–36,40]. Note that the stages of this procedure (that is, the transformation of the right hand side of the equation and the construction of Lyapunov functions) admit various implementations. This ambiguity can be advantageous, since by choosing various resulting functionals, one can substantially enlarge the stability domain.

3.8.2. Statement of the problem

Let Eq. (3.95) have the form
\[
\Delta x_n = x_{n+1} - x_n = -\sum_{k=0}^{\tau} b_{n,k}x_{n-k}, \quad n \geq 0,
\]
(3.95)

where \(n\) is discrete time, \(\tau\) is a given nonnegative number, and the constants \(b_{n,k}\) are given for \(n \geq 0\), \(0 \leq k \leq \tau\), and \(\tau < n\). For further considerations, it is convenient to define \(b_{n,k}\) also for \(k > \tau\) by setting \(b_{n,k} = 0\) for \(n \geq 0\) and \(k > \tau\). The initial conditions for Eq. (3.95) are prescribed for \(n = -\tau, \ldots, 0\).

Our problem is to study the asymptotic properties of solutions of equation (3.95) as \(n \to \infty\). We use the procedure for the construction of Lyapunov functionals for finite difference equations. First, we present the right-hand side of the original Eq. (3.95) as the sum
\[
\Delta x_n = -q_n x_n + \Delta F_{2n},
\]

(3.96)
where

\[ F_{2n} = \sum_{k=1}^{\tau} \sum_{s=1}^{k} b_{n+k-s,k} x_{n-s}, \]  
(3.97)

\[ q_n = \sum_{k=0}^{\tau} b_{n+k,k}. \]  
(3.98)

Let us show that (3.96) is equivalent to (3.95).

Indeed, by applying the operator \( p_{SOH} \) to relation (3.97), we obtain:

\[ \Delta F_{2n} = \sum_{k=1}^{\tau} \left[ \sum_{s=1}^{k} b_{n+1+k-s,k} x_{n+1-s} - \sum_{s=1}^{k} b_{n+k-s,k} x_{n-s} \right]. \]

Reducing the expression in brackets to the form \( b_{n+k,k} x_n - b_{n,k} x_{n-k} \) yields:

\[ \Delta F_{2n} = \sum_{k=1}^{\tau} \left[ b_{n+k,k} x_n - b_{n,k} x_{n-k} \right]. \]  
(3.99)

The substitution of (3.99) into (3.96) results in Eq. (3.95):

\[ \Delta F_{2n} - q_n x_n = \sum_{k=1}^{\tau} \left[ b_{n+k,k} x_n - b_{n,k} x_{n-k} \right] - \sum_{k=0}^{\tau} b_{n+k,k} x_n = - \sum_{k=0}^{\tau} b_{n,k} x_{n-k}. \]

3.8.3. Main results

We introduce the following notation:

\[ \bar{q}_n = \sum_{k=0}^{\tau} |b_{n+k,k}|, \]  
(3.100)

\[ \alpha = \sup_{n \geq 0} \left\{ -2 + q_n + \sum_{k=1}^{\tau} \sum_{s=1}^{k} |b_{n+k-s,k}| + M \sum_{j=1}^{\tau} q_{n+j} \right\}. \]  
(3.101)

The existence of a limit of the solution is guaranteed by the following statement.

**Theorem 3.9.** If for the finite difference equations (3.95) there exists a constant \( M \) such that

\[ \bar{q}_n \leq M q_n, \]  
(3.102)

and if the conditions \( q_n > 0 \) and \( \alpha < 0 \) are satisfied, then the solutions \( x_n \) of the finite difference equations have a limit as \( n \to \infty \).

**Proof.** For Eq. (3.96) we construct a functional satisfying conditions proposed in the papers [25,26].

Let us construct a Lyapunov function for the auxiliary equation \( \Delta y_n = y_{n+1} - y_n = -q_n y_n \) without aftereffect. Let \( v_n = y_n^2 \). Then an application of the operator \( \Delta \) to \( v_n \) yields \( \Delta v_n = \Delta y_n^2 = y_{n+1}^2 - y_n^2 = -q_n (2 - q_n) y_n \). If \( \sup_{n \geq 2} \{ q_n \} < 2 \), then, by virtue of
the assumption $\Delta v_n \leq 0$ the function $v_n = y_n^2$ is a Lyapunov function for the equation $\Delta y_n = -q_n y_n$.

Now we construct a Lyapunov functional $v_n$ for Eq. (3.96) in the form $V_n = V_{1n} + V_{2n}$, where $V_{1n} = (y_n) = (x_n - F_{2n})^2$. We use assumption (3.102) of Theorem 3.9. Then, taking into account the formula for $F_{2n}$, we obtain

$$
\Delta V_{1n} = \Delta y_n^2 = y_{n+1}^2 - y_n^2 = (y_{n+1} + y_n)(y_{n+1} - y_n)
$$

$$
= (2y_n - q_n x_n)(-q_n x_n) = q_n (q_n x_n^2 - 2x_n^2 + 2x_n F_{2n})
$$

$$
\leq q_n \left[ q_n x_n^2 - 2x_n^2 + x_n^2 \sum_{k=1}^{K} \sum_{s=1}^{K} |b_{n+k-s,k}| + \sum_{s=1}^{K} x_{n-s}^2 \sum_{k=s}^{K} |b_{n+k-s,k}| \right]
$$

$$
\leq q_n \left( -2 + q_n + \tau \sum_{k=1}^{K} \sum_{s=1}^{K} |b_{n+k-s,k}| \right) x_n^2 + q_n \sum_{s=1}^{K} x_{n-s}^2 \sum_{k=s}^{K} |b_{n+k-s,k}|.
$$

We choose the functional $V_{2n}$ in the form

$$
V_{2n} = \sum_{k=1}^{K} \sum_{j=1}^{j} \sum_{\mu=1}^{j} q_{n+k-j} \sum_{\mu=1}^{j} |b_{n+k-\mu,\mu}| x_{n-\mu}^2,
$$

then

$$
\Delta V_{2n} = \sum_{k=1}^{K} \left( \sum_{j=1}^{j} q_{n+1+k-j} \sum_{\mu=1}^{j} |b_{n+1+k-\mu,\mu}| x_{n+1-\mu}^2 \right)
$$

$$
- \sum_{j=1}^{j} q_{n+k-j} \sum_{\mu=1}^{j} |b_{n+k-\mu,\mu}| x_{n-\mu}^2.
$$

Since

$$
\sum_{j=1}^{j} q_{n+1+k-j} \sum_{\mu=1}^{j} |b_{n+1+k-\mu,\mu}| x_{n+1-\mu}^2 = \sum_{j=0}^{j} q_{n+k-j} \sum_{\mu=0}^{j} |b_{n+k-\mu,\mu}| x_{n-\mu}^2,
$$

we obtain

$$
\Delta V_{2n} = \sum_{k=1}^{K} \left[ \sum_{j=0}^{j} \sum_{\mu=0}^{j} q_{n+k-j} \sum_{\mu=0}^{j} |b_{n+k-\mu,\mu}| x_{n-\mu}^2 \right]
$$

$$
= \sum_{k=1}^{K} q_{n+k} |b_{n+k,k}| x_n^2 \sum_{j=1}^{j} q_{n+k-j} \sum_{\mu=0}^{j} |b_{n+k-\mu,\mu}| x_{n-\mu}^2.
$$
with regard for notation (3.102), yields

\[-q_n \sum_{\mu=1}^{k} |b_{n+k-\mu,k}|x_{n-\mu}^2 - \sum_{j=1}^{k-1} q_{n+k-j} \sum_{\mu=1}^{j} |b_{n+k-\mu,k}|x_{n-\mu}^2 \]

\[= \sum_{k=1}^{\tau} \left[ \sum_{j=0}^{k-1} q_{n+k-j} |b_{n+k,k}|x_n^2 - q_n \sum_{\mu=1}^{k} |b_{n+k-\mu,k}|x_{n-\mu}^2 \right].\]

Therefore, for \( V_n = V_{1n} + V_{2n} \) we have

\[\Delta V_n \leq q_n \left( -2 + q_n + \sum_{k=1}^{\tau} \sum_{s=1}^{k} |b_{n+k-k,s,k}| \right) x_n^2 + x_n^2 \sum_{k=1}^{\tau} \sum_{j=0}^{k-1} q_{n+k-j} |b_{n+k,k}|.\]

Since

\[\sum_{k=1}^{\tau} \sum_{j=0}^{k-1} |b_{n+k,k}|q_{n+k-j} \leq \sum_{k=0}^{\tau} \sum_{j=0}^{\tau-1} |b_{n+k,k}| \sum_{j=0}^{\tau-1} q_{n+j+1} = \bar{q}_n \sum_{j=1}^{\tau} q_{n+j} \leq q_n M \sum_{j=1}^{\tau} q_{n+j},\]

we have

\[\Delta V_n \leq q_n x_n^2 \left( -2 + q_n + \sum_{k=1}^{\tau} \sum_{s=1}^{k} |b_{n+k-k,s,k}| + M \sum_{j=1}^{\tau} q_{n+j} \right),\]

or, with regard for notation (3.101):

\[\Delta V_n \leq \alpha q_n x_n^2.\] (3.103)

Therefore, by virtue of the assumption \( \alpha < 0 \) made in Theorem 3.9, the functional \( V_n \) satisfies that \( c_n = \alpha q_n \) and \( p = 2 \). Then, there exists a limit \( \lim_{n \to \infty} V_n \), and moreover, \( \lim_{n \to \infty} c_n x_n^2 = 0 \), whence \( \lim_{n \to \infty} \alpha q_n x_n^2 = 0 \), i.e.

\[\lim_{n \to \infty} q_n x_n^2 = 0.\] (3.104)

This, together with (3.103), implies that \( V_n \) is a monotone nonincreasing convergent sequence, i.e., \( \lim_{n \to \infty} V_n = q \geq 0 \).

Next, we note that

\[V_{2n} \leq \left( \sum_{j=1}^{\tau} q_{n+\tau-j} \right) \sum_{k=1}^{\tau} \sum_{\mu=1}^{\tau} |b_{n+k-k,\mu}|x_{n-\mu}^2 = \left( \sum_{j=1}^{\tau} q_{n+\tau-j} \right) \sum_{\mu=1}^{\tau} x_{n-\mu}^2 \bar{q}_{n-\mu} \]

\[\leq M \left( \sum_{j=1}^{\tau} q_{n+\tau-j} \right) \sum_{\mu=1}^{\tau} q_{n-\mu} x_{n-\mu}^2.\]

The expression in parentheses is bounded [see (3.103)]; therefore, condition (3.103), with regard for notation (3.102), yields \( M \sum_{j=1}^{\tau} q_{n+\tau-j} = M \sum_{j=0}^{\tau-1} q_{n+j} \leq 2 \). Consequently, \( V_{2n} \leq 2 \sum_{\mu=1}^{\tau} q_{n-\mu} x_{n-\mu}^2 \). Thus, \( V_{2n} \to 0 \) as \( n \to \infty \). Hence \( \lim_{n \to \infty} |y_n| = \lim_{n \to \infty} \sqrt{V_n} = \sqrt{q} \).

Let us prove that the limit \( \lim_{n \to \infty} y_n \) exists. If \( q = 0 \), then \( \lim_{n \to \infty} y_n = 0 \). Let us consider the case \( q > 0 \). If all terms of the sequence \( y_n \) are of the same sign, then the
convergence of $|y_n|$ implies that of $y_n$; i.e., the divergence of $y_n$ is possible if the $y_n$ are of variable sign.

Let us consider the case in which the $y_n$ are of variable sign. Let us take an $\varepsilon$, $0 < \varepsilon < \sqrt{\theta}$; then there exists an $N$ such that:

$$0 < \sqrt{\theta} - \varepsilon < |y_n| < \sqrt{\theta} + \varepsilon \quad \text{for } n > N. \quad (3.105)$$

Let us now consider the following two sets of indices $n > N$: the set $I$ of indices $n$ such that $y_n > 0$ and the set $J$ of indices $n$ such that $y_n < 0$. Since inequality (3.105) is valid for $n > N$ and $y_n$ is a sequence of alternating signs, it follows that there is no index $n > N$ with $y_n = 0$ and both $I$ and $J$ are bounded. Let us now choose a sequence $s_n$ such that $s_n \in I$, $s_n + 1 \in J$, and $s_n \to \infty$.

Then for the auxiliary finite difference equation $\Delta y_{s_n} = y_{s_n+1} - y_{s_n} = -q_{s_n} x_{s_n}$, where $\sqrt{\theta} - \varepsilon < y_{s_n} < \sqrt{\theta} + \varepsilon$ and $\sqrt{\theta} - \varepsilon < -y_{s_n+1} < \sqrt{\theta} + \varepsilon$, we have $-\sqrt{\theta} - \varepsilon < y_{s_n+1} < -\sqrt{\theta} + \varepsilon$, whence $-2(\sqrt{\theta} + \varepsilon) < y_{s_n+1} - y_{s_n} < 2(-\sqrt{\theta} + \varepsilon) < 0$. Therefore:

$$0 < 2(\sqrt{\theta} - \varepsilon) < q_{s_n} x_{s_n} < 2(\sqrt{\theta} + \varepsilon), \quad (3.106)$$

which is impossible. Indeed, let us consider the sequence $x_{s_n}$. It either tends to zero or not. If $x_{s_n} \to 0$, then, by virtue of the boundedness of $q_{s_n}$, we arrive at a contradiction with (3.106). If $x_{s_n} \not\to 0$, then there exists a sequence $\{x_{n}\}_{n \geq 0}$ such that $|x_{z_n}| > \delta > 0$.

If this is the case, then it follows from the convergence $x_{n}^2 q_n \to 0$ and from the inequality $|x_{z_n}| > \delta > 0$ that $x_{z_n} q_n \to 0$, which contradicts inequality (3.106). Thus, $y_n$ is a convergent sequence, i.e., the limit $\lim_{n \to \infty} y_n$ exists.

Let us prove that the convergence of the sequence $y_n$ implies the convergence of the sequence $x_n$.

Note that the relation $\Delta y_n = -q_n x_n$ yields $y_n - \lim_{j \to \infty} y_j = \sum_{j=n}^{\infty} q_j x_j$. Since $\lim_{n \to \infty} [y_n - \lim_{j \to \infty} y_j] = 0$, we have $\sum_{j=n}^{\infty} q_j x_j < \infty$ (the series is convergent), i.e.

$$q_j x_j \to 0 \quad \text{as } j \to \infty. \quad (3.107)$$

Let us give another proof of the convergence of the sequence $x_n$. If $x_n \to 0$ as $n \to \infty$, then relation (3.107) holds. If $x_n \not\to 0$ as $n \to \infty$, then there exists a sequence of indices $j$ such that $|x_j| > \varepsilon > 0$. But $y_n = x_n - F_{2n}$, i.e., $x_n$ is convergent (since so is $y_n$) as $F_{2n} \to 0$. But, by virtue of (3.104), $q_j x_j^2 \to 0$, i.e., $q_j |x_j| \to 0$. Consequently, $q_j x_j \to 0$ (since $|x_j| > \varepsilon > 0$). Therefore, $F_{2n} \to 0$ as $n \to \infty$, since

$$|F_{2n}| \leq \sum_{s=1}^{\tau} |x_{n-s}| \sum_{k=0}^{\tau} |b_{n+k-s,k}| \leq M \sum_{s=1}^{\tau} q_{n-s} |x_{n-s}| \to 0.$$

The proof of Theorem 3.9 is complete. \[\square\]

**Theorem 3.10.** Let all assumptions of Theorem 3.9 be valid for Eq. (3.95), and, in addition, let

$$\sum_{n=0}^{\infty} q_n = \sum_{n=0}^{\infty} \sum_{k=0}^{\tau} b_{n+k,k} = \infty. \quad (3.108)$$
Then the limit of solutions $x_n$ of the finite difference equations (3.95) as $n \to \infty$ is zero.

**Proof.** We rewrite Eq. (3.96) in the form $A(x_n - F_{2n}) = -q_n x_n$. Let $\lim_{n \to \infty} x_n = \delta > 0$. We choose $N$ such that $x_n = 2$ for all $n \geq N$. Note that

$$\sum_{n=N}^{L} A(x_n - F_{2n}) = -\sum_{n=N}^{L} q_n x_n \leq -\frac{\delta}{2} \sum_{n=N}^{L} q_n \to \infty \quad \text{as} \quad L \to \infty.$$ 

But the left hand side of this relation is finite for all $L$, since $\sum_{n=N}^{L} A(x_n - F_{2n}) = x_L - F_{2n}(L) - x_N + F_{2n}(N)$. We have arrived at a contradiction. Consequently, $x_n \to 0$ as $n \to \infty$, i.e., $\lim_{n \to \infty} x_n = 0$.

The proof of Theorem 3.10 is complete.

Sufficient conditions for the stability of solutions of Eqs. (3.95) are given by the following theorem.

**Theorem 3.11.** If the assumptions $q_n > 0$ and $z < 0$ of Theorem 3.9 are valid, relation (3.108) of Theorem 3.10 holds for Eq. (3.95), and

$$\gamma = \sup_{n \geq 0} \left\{ \sum_{k=1}^{\tau} \sum_{s=1}^{k} |b_{n+k-s,k}| \right\} < 1,$$

(3.109)

then the trivial solution of the linear finite difference equations (3.95) is stable.

**Proof.** Let all assumptions of the theorem be valid. Let us show that in this case the trivial solution of Eq. (3.95) is stable. We choose an arbitrary number $N > 0$. Then it follows from (3.95) that for all $n \leq N$:

$$V_0 \geq V_n \Rightarrow |x_n - F_{2n}| \leq \sqrt{V_0} \Rightarrow |x_n| \leq \sqrt{V_0 + |F_{2n}|} \Rightarrow |x_n| \leq \sqrt{V_0 + \left( \max_{n \leq N} |x_n| \right)^2} \Rightarrow \max_{n \leq N} |x_n| \leq \sqrt{V_0 + \gamma \max_{n \leq N} |x_n|} \Rightarrow \max_{n \leq N} |x_n| \leq \frac{\sqrt{V_0}}{(1 - \gamma)},$$

and we have $\max_{n \leq N} |x_n| \leq \sqrt{V_0}/(1 - \gamma) \leq C \max_{-\gamma \leq n \leq 0} |x_n|$ for some $C > 0$. The resultant inequality implies the stability of the trivial solution of equation (3.95) if we set $\epsilon = \delta$, where $\delta = C \max_{-\gamma \leq n \leq 0} |x_n|$. The proof of Theorem 3.11 is complete.

Let us give sufficient conditions for the asymptotic stability of solutions of equation (3.95).

**Theorem 3.12.** If all assumptions of Theorems 3.9–3.11 for equations of the form (3.95) are valid for all $n \geq 0$, the trivial solution of the linear finite difference equation (3.95) is asymptotically stable.

The proof of Theorem 3.12 follows from Theorems 3.9–3.11.
As a result, we see that our conditions for the asymptotic stability of nonstationary finite difference equations (3.95) depend only on the coefficients of the equations themselves.

**Example 3.5.** We set $\tau = 0$ in Eq. (3.95); then this equation can be written as

$$x_{n+1} = x_n - b_n x_n = x_n(1 - b_n).$$

The conditions $q_n > 0$ and $\varepsilon < 0$ imply that the coefficients $b_n$ must satisfy the inequalities

$$0 < b_n, \quad \sup_{n \geq 0} \{b_n\} < 2,$$

and condition (3.108) acquires the form

$$\sum_{i=0}^{\infty} b_i = \infty.$$

If $b_n \equiv b = \text{const}$, then the solution of (3.110) can be written as $x_n = (1 - b)^n x_0$. Hence the condition $0 < b < 2$ of the asymptotic stability coincides with the necessary and sufficient condition.

### 4. Boundedness of the solutions for Volterra difference equations

#### 4.1. Statement of the problem

In this paragraph we consider the boundedness of the solutions for Volterra equation

$$y_{n+1} = F(n, y_{n_0}, \ldots, y_n), \quad n \geq n_0, \quad y_{n_0} = y_0.$$  (4.1)

Here the discrete time $n$ and the initial time moment belong to the set of integers $\mathbb{N}_0$, the vector $y_n$ belongs to the real $r$-dimensional Euclidean space $\mathbb{R}^r$ with the norm $\| \cdot \|$, the function $F$ is defined as $F : N_0 \times S \to \mathbb{R}^r$, where $S$ is a space of sequences with elements from $\mathbb{R}^r$ and initial value $y_{n_0} = y_0$ is prescribed. It is assumed that for every $n \geq n_0$ function $F(n, y_{n_0}, \ldots, y_n)$ is defined only by the values $y_{n_0}, \ldots, y_n$ and does not depend on $y_j$ for $j > n$. Sometimes, in order to stress the dependence of the solution of problem (4.1) on initial time $n_0$ and initial position $y_0$, it will be denoted by $y(n, n_0, y_0)$, $n \geq n_0$. Different definitions for boundedness of solutions in (4.1) can be obtained under appropriate modifications of corresponding definitions for boundedness of solutions in differential equations. Consider two such definitions used in what follows.

**Definition 4.1.** System (4.1) is called:

1. bounded if for any $n_0 \in \mathbb{N}_0$ and number $m > 0$ there exists a number $\varepsilon(n_0, m)$ depending on $n_0$ and $m$ such that $\|y(n, n_0, y_0)\| < \varepsilon(n_0, m)$ for all $n \geq n_0$ and $y_0$, $|y_0| \leq m$;
uniformly bounded with respect to the initial moment \( n_0 \) if \( \varpi(n_0, m) \equiv \varpi(m) \), i.e., the constant bounding the solution does not depend on the initial moment \( n_0 \).

Conditions of boundedness and uniform boundedness for general equation (4.1) are obtained in [27]. Some of these conditions, depending on the existence of Lyapunov functionals, are not only sufficient, but also necessary.

Along with the applications of general Lyapunov method, we can use comparison principle and concrete equations to obtain boundedness of their solutions.

4.2. Conditions of boundedness

Before proving some conditions for boundedness and uniform boundedness of Eq. (4.1) we consider an example showing that a bounded system is not necessarily uniformly bounded.

**Example 4.1.** Consider the two-dimensional difference system

\[
y_{n+1} = \begin{pmatrix} 1 & (n + 2)(n + 3)^{-1} \\ 0 & (n + 2)^3(n + 3)^{-3} \end{pmatrix} y_n + \begin{pmatrix} 0 & (n + 1)^{-2} \\ 0 & 0 \end{pmatrix} y_{n_0}, \quad y_{n_0} = y_0.
\] (4.2)

The solution of Eq. (4.2) can be represented in the form

\[
y_n = R(n,n_0) y_0 + \sum_{l=n_0}^n R(n,l) g(l - 1), \quad n > n_0,
\] (4.3)

where the matrix \( R = (r_{n,j})_{n,j=1,2} \) and vector \( g \) are

\[
R(n,n_0) = \begin{pmatrix} 1 & -(n_0 + 1)(n_0 + 2)^2 + (n_0 + 2)^3(n + 1)(n + 2)^{-1} \\ 0 & (n_0 + 2)^3(n + 2)^{-3} \end{pmatrix},
\]

\[
g(n) = \begin{pmatrix} 0 & (n + 1)^{-2} \\ 0 & 0 \end{pmatrix} y_0.
\] (4.4)

Denote the components of the vector \( y_0 \) by \( y_{01} \) and \( y_{02} \). Then, by virtue of (4.4) we get

\[
\sum_{l=n_0}^n R(n,l) g(l - 1) = y_{02} \sum_{l=n_0+1}^n \begin{pmatrix} l^{-2} \\ 0 \end{pmatrix} \leq \pi^2 y_{02} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\] (4.5)

Hence the second addend at the right hand side of equality (4.3) is uniformly bounded with respect to \( n_0 \). At the same time the first addend at right hand side of equality (4.3) is unbounded in \( n_0 \). Really for the component \( r_{n2}(kn_0,n_0) \), for any integer \( k \),
we have
\[ r_{n,2}(kn_0, n_0) = \frac{(n_0 + 2)^2}{kn_0 + 2} [ - (n_0 + 1)(kn_0 + 2) + (n_0 + 2)(kn_0 + 1)] \]
\[ = n_0(n_0 + 2)(kn_0 - 1)(kn_0 + 2)^{-1} \to \infty, \quad n_0 \to \infty. \]

Thus any solution of equation (4.2) is bounded, with respect to \( n \) for arbitrary fixed \( n_0 \), because of (4.5) and boundedness of the matrix \( R(n, n_0) \). But relations (4.3) and (4.5) show that system (4.2) is bounded nonuniformly with respect to the initial moment \( n_0 \).

### 4.3. Boundedness results by the use of Lyapunov functional

Now let us consider boundedness theorems in terms of auxiliary Lyapunov functionals

\[ V : N_0 \times S \to \mathbb{R}, \]

such that \( V(n, y_{n_0}, \ldots, y_n, \ldots) = V(n, y_{n_0}, \ldots, y_n) \).

Let \( \omega_j : \mathbb{R}^+ \to \mathbb{R}^+ \), \( \omega_j(0) = 0 \) be continuous nondecreasing functions. A function \( V(n_0, y_{n_0}) \) will be called bounded with respect to the second argument if it is bounded for any fixed \( n_0 \) and any \( y_{n_0} \in \mathbb{R}^r \) with \( \|y_{n_0}\| < \infty \). Then the following result holds:

**Theorem 4.1.** Assume that there exist functions \( V(n, y_{n_0}, \ldots, y_n) \) and \( \omega_1 \) such that

\[ \omega_1(\|y_n\|) \leq V(n, y_{n_0}, \ldots, y_n), \quad n \geq n_0, \quad (4.6) \]

and

\[ \Delta V = V(n + 1, y_{n_0}, \ldots, y_n, F(n, y_{n_0}, \ldots, y_n)) - V(n, y_{n_0}, \ldots, y_n) \leq 0, \quad (4.7) \]

where the function \( V(n_0, y_{n_0}) \) is bounded with respect to the second argument on any compact set of variation \( y_{n_0} \) and

\[ \omega_1(s) \to \infty \quad \text{as} \quad s \to \infty. \quad (4.8) \]

Then system (4.1) is bounded.

**Proof.** Take any \( s_0 \) and consider arbitrary \( y_{n_0}, \|y_{n_0}\| \leq s_0 \). From conditions (4.6) and (4.7) it follows that

\[ \omega_1(\|y_n\|) \leq V(n, y_0, \ldots, y_n) \leq V(n_0, y_0). \quad (4.9) \]

Boundedness of the function \( V(n_0, y_0) \) and assumption (4.8) mean the existence of the constant \( \varepsilon(n_0, s_0) \) such that

\[ V(n_0, y_{n_0}) \leq \omega_1(\varepsilon(n_0, s_0)). \]

Consequently taking into account (4.9) we get the estimate

\[ \omega_1(\|y_n\|) \leq \omega_1(\varepsilon(n_0, s_0)). \]

Thus \( \|y(n, n_0, y_0)\| \leq \varepsilon(n_0, s_0) \) and the theorem is proven. \( \square \)

It is a natural question to ask and certainly it would be useful to specify the boundedness conditions under which the appropriate Lyapunov functional does exist. Therefore
in the next theorem we prove the conditions (4.6)–(4.8) are not only sufficient but also necessary boundedness conditions.

**Theorem 4.2.** Assume that system (4.1) is bounded, then a Lyapunov functional \( V(n, y_{n_0}, \ldots, y_n) \) and a function \( \omega_1 \) satisfying conditions (4.6)–(4.8) exists.

**Proof.** Take any fixed integer \( n_0 \leq n \). Assume that the solution of problem (4.1); \( \{y_{n_0}, y_{n_0+1}, \ldots, y_{1}\} \) is found on the set of integers \( j : n_0 \leq j \leq n \). Then for \( n \geq n_0 \) the solution of problem (4.1) is a function \( y(n; \tau; y_0, \ldots, y_{1}) \) depending on the choice of a new initial time moment \( \tau \) and new initial conditions \( (y_0, \ldots, y_{1}) \). From this interpretation it follows that

\[
y(n; n_0; y_0) \equiv y(n; \tau; y_0, \ldots, y_{1}), \quad n_0 \leq \tau \leq n. \quad (4.10)
\]

Now, using the boundedness of solutions of system (4.1), we introduce the functional:

\[
V(n, y_0, \ldots, y_n) = \sup_{j \geq 0} \{\| y(n + j, n, y_0, \ldots, y_n) \| \}, \quad (4.11)
\]

which satisfies inequality (4.6) with \( \omega_1(\| y_n \|) = \| y_n \| \). Hence, relation (4.8) is also valid and due to (4.11) the function \( V(n_0, y_{n_0}) \) is bounded with respect to the second argument.

Now, in order to prove inequality (4.7), take into account that

\[
V(n + 1, y_0, \ldots, y_{n+1}) = \sup_{j \geq 0} \{\| y(n + 1 + j, n + 1, y_0, \ldots, y_{n+1}) \| \}
\]

\[
= \sup_{j \geq 0} \{\| y(n + 1 + j, n_0, y_0) \| \} = \sup_{j \geq 1} \{\| y(n + j, n_0; y_0) \| \}
\]

\[
\leq \sup_{j \geq 0} \{\| y(n + j, n_0; y_0) \| \} = \sup_{j \geq 0} \{\| y(n + j; n; y_0, \ldots, y_n) \| \}
\]

\[
= V(n, y_0, \ldots, y_n).
\]

Consequently inequality (4.7) is also valid and the necessity of the conditions in Theorem 4.2 is proven.

**Theorem 4.3.** Let relations (4.6)–(4.9) be valid and there exists the function \( \omega_2 \) such that

\[
V(n_0, y_{n_0}) \leq \omega_2(\| y_{n_0} \|). \quad (4.12)
\]

Then system (4.1) is uniformly bounded.

**Proof.** Take any \( s_0 \) and consider \( y_{n_0} : \| y_{n_0} \| < s_0 \). Then from (4.6), (4.7), and hypotheses (4.12), we obtain

\[
\omega_1(\| y_n \|) \leq V(n, y_0, \ldots, y_n) \leq V(n_0, y_0) \leq \omega_2(\| y_0 \|) \leq \omega_2(s_0).
\]

Further in view of (4.8) there is a number \( x(s_0) \) for which \( \omega_1(x(s_0)) \geq \omega_2(s_0) \). From here and (4.13) it follows that \( \omega_1(\| y_n \|) \leq \omega_1(x(s_0)) \). Hence \( \| y(n, n_0, y_0) \| \leq x(s_0) \) for all \( n_0, n \geq n_0, \| y_0 \| \leq s_0 \) and the theorem is proven.
Thus, in order to obtain conditions of boundedness of concrete equations using the above theorems, the construction of appropriate Lyapunov functions is required. This can be realized by means of the general procedure for the construction of Lyapunov functions for stability problems [25].

Let us illustrate this assertion by an example showing the form of the functionals $V$ and the corresponding boundedness conditions.

Example 4.2. Consider the scalar equation:

$$y_{n+1} = a_n y_n + b_n y_n \left[ 1 + \sum_{j=0}^{n-1} y_j^2 \right]^{-1}, \quad n > 0.$$  \hspace{1cm} (4.14)

Take $V = \|y_n\|$. Then we get

$$\Delta V = |y_{n+1}| - |y_n| \leq (-1 + |a_n|)|y_n| + |b_n y_n| \left[ 1 + \sum_{j=0}^{n-1} y_j^2 \right]^{-1} \leq (-1 + |a_n| + |b_n|)|y_n|.$$  

So, according to Theorem 4.3 the boundedness condition for Eq. (4.14) takes the form $|a_n| + |b_n| \leq 1$.

Now we want to prove that for some particular classes of Volterra equations the assumptions of Theorems 4.1 and 4.3 can be weakened.

**Theorem 4.4.** Assume that the right hand side of Eq. (4.1) satisfies inequalities

$$\|F(n, y_{n_0}, \ldots, y_n)\| \leq \sum_{j=0}^{n-n_0} a_{n,j} \|y_{n-j}\| + b_n, \quad n \geq n_0,$$  \hspace{1cm} (4.15)

with $a_{n,j}, b_n$ nonnegative numbers and that

$$a = 1 - \sup_{n \geq n_0} \left\{ \sum_{j=n_0}^{\infty} a_{n+j,j} \right\} > 0, \quad b = \sum_{j=n_0}^{\infty} |b_j| < \infty.$$  \hspace{1cm} (4.16)

Then system (4.1) is uniformly bounded.

**Proof.** According to the general method for construction of Lyapunov functions, let us consider the function

$$V(n, y_{n_0}, \ldots, y_n) = V_1 + V_2,$$

with

$$V_1 = \|y_n\|, \quad V_2 = \sum_{l=0}^{n-n_0} \|y_{n-l}\| \sum_{j=l}^{\infty} a_{n+j-l,j}.$$  \hspace{1cm} (4.17)
By virtue of (4.15)–(4.17), we have
\[
\Delta V = \|y_{n-1}\| + \sum_{l=1}^{n+1-n_0} \|y_{n+1-l}\| \sum_{j=l}^{\infty} a_{n+1+j-l,j} - \|y_n\|
- \sum_{l=1}^{n-n_0} \|y_{n-l}\| \sum_{j=l}^{\infty} a_{n+j-l,j} \leq -a\|y_n\| + b_n.
\] (4.18)

Assume now that Eq. (4.1) has unbounded solution \(\{y_n\}_{n \geq 0}\). Then, by virtue of (4.17), the function \(V\) is also unbounded. It means existence of the subsequence of indexes \(\{pFS_n\}_{n \geq 0}\); \(pFS_n \to \infty\), as \(n \to \infty\), such that for all \(n_0 \leq j \leq \tau_n\):
\[
V(j, y_0, \ldots, y_j) \leq V(\tau_n, y_0, \ldots, y_{\tau_n}).\] (4.19)

Now take an arbitrary moment \(\tau\) belonging to the subsequence \(\{\tau_n\}_{n \geq 0}\) and estimate \(\|y_n\|\) for \(n_0 \leq n \leq \tau\). First of all, by virtue of (4.19), we get
\[
V(\tau - 1, y_0, \ldots, y_{\tau-1}) \leq V(\tau, y_0, \ldots, y_{\tau}).
\]
From here and (4.18) it follows that
\[
0 \leq V(\tau, y_0, \ldots, y_{\tau}) - V(\tau - 1, y_0, \ldots, y_{\tau-1}) \leq -a\|y_{\tau-1}\| + b_{\tau-1}.
\]
Hence:
\[
\|y_{\tau-1}\| \leq \frac{1}{a} b_{\tau-1}.
\]
Further, taking into account the definition of the function \(V\) and relations (4.17) and (4.18) we have
\[
V(\tau, y_0, \ldots, y_{\tau}) \leq V(\tau - 1, y_0, \ldots, y_{\tau-1}) + b_{\tau-1}
= \|y_{\tau-1}\| + \sum_{l=1}^{\tau-1-n_0} \|y_{\tau-1-l}\| \sum_{j=l}^{\infty} a_{\tau-1-l+j,j} + b_{\tau-1}
\leq \left(1 + \frac{1}{a}\right) b_{\tau-1} + (1 - a) \sum_{l=1}^{\tau-1-n_0} \|y_{\tau-1-l}\|.
\] (4.20)

In order to estimate the sum at the right hand side of inequality (4.20) let us take into account that, by virtue of (4.18):
\[
V(\tau, y_0, \ldots, y_{\tau}) - V(n_0, y_0) \leq -a \sum_{l=n_0}^{\tau-1} \|y_l\| + \sum_{l=n_0}^{\tau-1} b_l.
\]
Consequently,
\[
\sum_{l=n_0}^{\tau-1} \|y_l\| \leq \frac{1}{a} \left(V(n_0, y_0) + \sum_{l=n_0}^{\tau-1} b_l\right) \leq \frac{1}{a} (V(n_0, y_0) + b).
\]
So because of (4.20):
\[ V(\tau, y_0, \ldots, y_{\tau}) \leq \left( 1 + \frac{1}{a} \right) b_{\tau-1} + \frac{1-a}{a} (V(n_0, y_0) + b). \] (4.21)

But, in view of (4.17) and (4.19) for any \( j \leq \tau \) the following inequality is valid:
\[ \|y_j\| \leq V(j, y_0, \ldots, y_{j}) \leq V(\tau, y_0, \ldots, y_{\tau}), \quad j \leq \tau. \]

From here and (4.21) it follows that:
\[ \|y_j\| \leq V(\tau, y_0, \ldots, y_{\tau}). \]

Finally the arbitrariness of \( \tau \in \{\tau_n\}_{n \geq 0} \) and the relation \( \tau_n \to \infty \) as \( n \to \infty \) imply that the sequence \( \{\|y_n\|\}_{n \geq 0} \) is uniformly bounded.

4.4. Boundedness result by use of the comparison theorem

In addition to the direct Lyapunov method, boundedness of solutions of some classes of Volterra equations can be investigated by other ways. Here we will describe comparison method, for the equation:
\[ M \sigma^{\infty} \begin{bmatrix} \dot{x}_n \\ x_{n+1} - x_n = F(n, y_0, \ldots, y_n) \\ y_{n+1} = y_0 \\ y_0 = x_0 \end{bmatrix}, \quad n \geq n_0, \quad y_{n_0} = y_0. \] (4.22)

Here, \( y_n \in \mathbb{R}^r \) and the function \( F: N_0 \times S \to \mathbb{R}^r \) satisfies the conditions:
\[ \|F(n, y_0, \ldots, y_n)\| \leq g(n, \|y_0\|, \ldots, \|y_n\|), \] (4.23)

where the scalar function \( g: N_0 \times \mathbb{R}^+ \to \mathbb{R}^+ \) is nondecreasing with respect to \( |y_j|, \ 0 \leq j \leq n \). Consider also the scalar equation:
\[ \Delta x_n = g(n, x_0, \ldots, x_n), \quad n \geq n_0, \quad x_{n_0} = x_0. \] (4.24)

**Theorem 4.5.** Assume that \( \|y_0\| \leq x_0 \) and all the assumptions made above are valid. Then if the solutions \( x_n \) of the problem (4.24) is bounded then the solution \( y_n \) of problem (4.22) is also bounded and the limit of the sequence \( \{y_n\}_{n \geq 0} \) as \( n \to \infty \) exists. Moreover if system (4.24) is uniformly bounded then (4.22) is also uniformly bounded.

**Proof.** From Eq. (4.22) and by virtue of (4.23), it follows that:
\[ \|y_n\| \leq \|y_0\| + \sum_{j=n_0}^{n-1} \|F(j, y_0, \ldots, y_j)\| \leq \|y_0\| + \sum_{j=n_0}^{n-1} g(j, \|y_0\|, \ldots, \|y_j\|). \] (4.25)

Moreover taking into account (4.24) we get
\[ x_n = x_0 + \sum_{j=n_0}^{n-1} g(j, x_0, \ldots, x_j). \] (4.26)

From relations (4.25) and (4.26), by using mathematical induction we deduce that \( \|y_n\| \leq x_n \). Hence system (4.22) is uniformly bounded. Further, since \( \{x_n\}_{n \geq 0} \) is
bounded and nondecreasing, it follows that \( \lim_{n \to \infty} x_n \) exists. So
\[
\|y_n - y_m\| \leq \sum_{j=m}^{n-1} \|F(j, y_0, \ldots, y_j)\| \leq \sum_{j=m}^{n-1} g(j, \|y_0\|, \ldots, \|y_j\|) \\
\leq \sum_{j=m}^{n-1} g(j, \|x_0\|, \ldots, \|x_j\|) = x_n - x_m \to 0, \quad n, m \to \infty.
\]
Thus the sequence \( \{y_n\}_{n \geq 0} \) is fundamental, i.e., there exists the limit of it as \( n \to \infty \). Theorem 4.5 is proven. \( \square \)

The following example shows that the more weak hypothesis \( \lim_{j \to \infty} \|B_{n,j}\| = 0, \ n > j \), is not sufficient to assure the boundedness of Eq. (4.1).

**Example 4.3.** Consider for \( n > 0 \) the two-dimensional perturbed system:
\[
x_{n+1} = x_0 \cos\left(\frac{\pi}{2}(n + 1)\right) + y_0 \sin\left(\frac{\pi}{2}(n + 1)\right) \\
- 2 \sum_{j=0}^{n} \frac{x_j}{j+1} \sin\left(\frac{\pi}{2}(n - j)\right),
\]
\[
y_{n+1} = -x_0 \sin\left(\frac{\pi}{2}(n + 1)\right) + y_0 \cos\left(\frac{\pi}{2}(n + 1)\right) \\
- 2 \sum_{j=0}^{n} \frac{x_j}{j+1} \cos\left(\frac{\pi}{2}(n - j)\right).
\]
(4.27)

The nonperturbed equation corresponding to Eq. (4.27) has the form
\[
x_{n+1} = x_0 \cos\left(\frac{\pi}{2}(n + 1)\right) + y_0 \sin\left(\frac{\pi}{2}(n + 1)\right),
\]
\[
y_{n+1} = -x_0 \sin\left(\frac{\pi}{2}(n + 1)\right) + y_0 \cos\left(\frac{\pi}{2}(n + 1)\right), \quad n \geq 0.
\]
(4.28)

From the last relations it follows that the nonperturbed system is uniformly bounded. At the same time any nonzero solution of the perturbed system (4.27) is unbounded because it may be written in the form
\[
x_n = x_0 \cos\left(\frac{\pi}{2} n\right) + y_0 \sin\left(\frac{\pi}{2} n\right) - 2 \sum_{j=0}^{n-1} \left[\sin\left(\frac{\pi}{2}(n - 1 - j)\right)\right] \\
\times \left[\left(\cos\left(\frac{\pi}{2} j\right)\right) x_0 + \left(\sin\left(\frac{\pi}{2} j\right)\right) y_0\right],
\]
\[ y_n = -x_0 \sin \left( \frac{\pi}{2} n \right) + y_0 \cos \left( \frac{\pi}{2} n \right) - 2 \sum_{j=0}^{n-1} \left[ \cos \left( \frac{\pi}{2} (n - 1 - j) \right) \right] \]

\times \left[ \left( \cos \left( \frac{\pi}{2} j \right) \right) x_0 + \left( \sin \left( \frac{\pi}{2} j \right) \right) y_0 \right].

Consequently

\[ x_n = \left( 2 \sum_{j=0}^{n-1} \cos^2 \left( \frac{\pi}{2} j \right) + 1 \right) \left( \cos \left( \frac{\pi}{2} n \right) \right) x_0 \]

\[ + \left( 2 \sum_{j=0}^{n-1} \sin^2 \left( \frac{\pi}{2} j \right) + 1 \right) \left( \sin \left( \frac{\pi}{2} n \right) \right) y_0 \]

\[ = (n + 1) \left[ x_0 \cos \left( \frac{\pi}{2} n \right) + y_0 \sin \left( \frac{\pi}{2} n \right) \right]. \]

In a similar way, we get

\[ y_n = (n + 2) \left[ -x_0 \sin \left( \frac{\pi}{2} n \right) + y_0 \cos \left( \frac{\pi}{2} n \right) \right]. \quad (4.29) \]

Hence, if \( x_0 \neq 0 \) or \( y_0 \neq 0 \) then the solutions of Eq. (4.27) are unbounded. In addition matrix \( B_{n,j} \) has the form

\[ B_{n,j} = \begin{pmatrix} \sin \left( \frac{\pi}{2} (n - j) \right) & 0 \\ \cos \left( \frac{\pi}{2} (n - j) \right) & 0 \end{pmatrix}. \]

Therefore \( \|B_{n,j}\| \to 0 \) as \( j \to \infty \).

4.5. Estimate os solution for nonlinear Volterra equations

Volterra equations, whose solution is defined by the whole previous history, are widely used in the modelling of the processes in continuous mechanics and biomechanics, problems of control and estimation and also some schemes of numerical solutions of integral and integral-differential equations with continuous time.

In this connection essential interest represents such properties of the solutions as stability, limiting periodicity, boundedness and various estimates of the solutions defined by the acting perturbations.

In [9] such estimates were obtained for the solutions of nonlinear scalar integral Volterra equations of convolution type:

\[ x(t) = f(t) + \int_0^t a(t - s)g(x(s)) \, ds, \quad t \geq 0. \quad (4.30) \]

Here perturbation \( f(t) \) is a function of bounded variation on \([0, \infty)\). Constructions of the paper [58] were founded on the accurate calculation of the positive and negative parts of the function \( g(x) \) incoming in the above mentioned integral. Under some
assumptions with respect to the kernel $a(t)$ the estimates of the solutions do not depend on the kernel $a(t)$ and defined only by the properties of the perturbation $f(t)$.

The estimates of the paper [58] were improved in [64] for scalar linear and nonlinear Volterra integral equations.

In this paper we consider Volterra equations with discrete time. The estimates of the solutions for both linear and nonlinear equations are derived, using some comparison theorems from the monograph [62] and auxiliary formula for representations of the solutions.

Let us consider Volterra scalar nonlinear difference equation

$$x_n = \sum_{j=1}^{n} F(n,j,x_j) + f_n, \quad n \geq 1. \quad (4.31)$$

**Theorem 4.6.** Let us assume that in Eq. (4.31) $f_n$ is a given sequence satisfying condition:

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty, \quad (4.32)$$

the function $F(n,j,x)$ is a continuous function with respect to the third argument, and moreover the function $xF(n,j,x)$ is nonpositive for all $x \in (-\infty, \infty)$ and nonincreasing with respect to $n$, $n \geq j$, for all $x \in (-\infty, \infty)$. Then the solutions $x_n$ of Eq. (4.32) satisfy the inequalities which do not depend on $F$:

$$f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \leq 2x_n \leq f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j|, \quad n \geq 1. \quad (4.33)$$

It is assumed that in (4.33) both sums are equal zero for $n = 1$.

**Proof.** Let us introduce the function $u_n$, equal:

$$u_n = \sum_{j=1}^{n} F_+(n,j,x_j) + \frac{1}{2} \left( f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) - \varepsilon n$$

$$= \sum_{j=1}^{n} F_+(j,j,x_j) + \sum_{j=1}^{n} \left[ F_+(n,j,x_j) - F_+(j,j,x_j) \right]$$

$$+ \frac{1}{2} \left( f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) - \varepsilon n.$$

Here $F_+ = \max(0,F)$ and $\varepsilon > 0$ is some number. First of all let us show that the function

$$F_+(n,j,x_j) - F_+(j,j,x_j),$$
is nonincreasing with respect to $n$, $n \geq j$, and nonpositive. Because of the inequality $xF(n, j, x) \leq 0$ for all $x \in (-\infty, \infty)$ we have

$$F(n, j, x) \geq 0 \quad \text{for } x \leq 0,$$

$$F(n, j, x) \leq 0 \quad \text{for } x \geq 0.$$

(4.34)

Further by virtue of the conditions of Theorem 4.6:

$$x[F(n + 1, j, x) - F(n, j, x)] \geq 0, \quad x \in (-\infty, \infty).$$

Therefore

$$F(n + 1, j, x) - F(n, j, x) \leq 0 \quad \text{for } x \leq 0,$$

$$F(n + 1, j, x) - F(n, j, x) > 0 \quad \text{for } x > 0,$$

From here and (4.34) it follows that

$$F_+(l + 1, j, x) - F_+(l, j, x) \leq 0, \quad x \leq 0,$$

$$F_+(l + 1, j, x) - F_+(l, j, x) = 0, \quad x \geq 0, \quad l \geq j.$$

Let us summarize both parts of the last relations with respect to $l$ from $l=j$ to $l=n-1$. As a result we obtain:

$$F_+(n, j, x) - F_+(j, j, x) \leq 0, \quad x \leq 0,$$

$$F_+(n, j, x) - F_+(j, j, x) = 0, \quad x \geq 0, \quad n \geq j.$$ (4.35)

Hence the sum

$$\sum_{j=1}^{n} (F_+(n, j, x_j) - F_+(j, j, x_j))$$

(4.36)

is nonincreasing with respect to $n$ and also nonpositive by virtue of (4.35). Let us show further that $u_1 < 0$. From the definition of the function $u_n$ we get:

$$u_1 = F_+(1, 1, x_1) + \frac{1}{2} (f_1 - |f_1|) - \varepsilon.$$

Suppose firstly that $f_1 > 0$. Then from the properties (4.34) of the function $F(n, j, x)$ any root of the equation:

$$x_1 = F(1, 1, x_1) + f_1,$$

must satisfy the condition $0 \leq x_1 \leq f_1$. Hence, $F_+(1, 1, x_1) = 0$. Consequently $u_1 = -\varepsilon < 0$ for $f_1 > 0$. If $f_1 < 0$, then any root of the equation $x_1 = F(1, 1, x_1) + f_1$ must satisfy the condition $f_1 \leq x_1 \leq 0$. Hence, $F(1, 1, x_1) = F_+(1, 1, x_1)$ and also $-f_1 \geq F(1, 1, x_1) \geq 0$. So, in this case:

$$u_1 \leq -f_1 + \frac{1}{2} (f_1 - |f_1|) - \varepsilon = -\varepsilon < 0,$$
for $f_1 < 0$. At last, if $f_1 = f_2 = \cdots = f_m = 0$, then it would be sufficient to come to the first nonzero number $f_j$.

Let us introduce one more function $v_n$ by the relation:

\[
v_n = -\sum_{j=1}^{n} F_-(j,j,x_j) - \sum_{j=1}^{n} [F_-(n,j,x_j) - F_-(j,j,x_j)]
\]

\[- \frac{1}{2} \left( f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) - \varepsilon n, \quad n \geq 1, \quad \varepsilon > 0.
\]

Here $F_- = \min(0,F)$. Let us check that $v_1 < 0$, using the same arguments as above for the proof that $u_1 < 0$. We have:

\[
v_1 = -F_-(1,1,x_1) - \frac{1}{2}(f_1 + |f_1|) - \varepsilon.
\]

If $f_1 > 0$ then any root of the equation $x_1 = F(1,1,x_1) + f_1$ must satisfy the condition $0 \leq x_1 \leq f_1$. Hence, $F_-(1,1,x_1) = F(1,1,x_1)$ and moreover $-f_1 \leq F_-(1,1,x_1) \leq 0$, i.e. $0 \leq -F_-(1,1,x_1) \leq f_1$. Therefore:

\[
v_1 \leq f_1 - \frac{1}{2}(f_1 + |f_1|) - \varepsilon = -\varepsilon < 0.
\]

If $f_1 < 0$, then any root of the equation:

\[x_1 = F(1,1,x_1) + f_1,
\]

will satisfy the condition $f_1 \leq x_1 \leq 0$, i.e., $F_-(1,1,x_1) = 0$. It means that $v_1 = -\varepsilon < 0$. As a result for all cases $v_1 = -\varepsilon < 0$. Let us show that the function

\[F_-(n,j,x_j) - F_-(j,j,x_j),
\]

as a function of $n$ is nondecreasing. Since the function $xF(n,j,x)$ is nondecreasing for all $x \in (-\infty, \infty)$, we can conclude that

\[x[F(n+1,j,x) - F(n,j,x)] \geq 0.
\]

Therefore

\[F(n+1,j,x) - F(n,j,x) \geq 0, \quad x \geq 0.
\]

Besides

\[F_-(n,j,x) = F(n,j,x), \quad x \geq 0.
\]

It means that

\[F_-(n+1,j,x) - F_-(n,j,x) \geq 0.
\]

From here it follows that the function

\[F_-(n,j,x) - F_-(j,j,x), \quad n \geq j,
\]
is nonnegative and nondecreasing function of $n$, $n \geq j$. Hence the sum

$$
\sum_{j=1}^{n} [F_-(n,j,x) - F_-(j,j,x)],
$$

(4.37)

is nondecreasing with respect to $n$ and nonnegative. Let us remark also that from the definitions of the functions $u_n, v_n$ and Eq. (4.31) it follows that

$$
x_n = \sum_{j=1}^{n} [F_+(n,j,x_j) + F_-(n,j,x_j)] + f_n = u_n - v_n.
$$

(4.38)

Further, because the sum (4.36) is nonincreasing with respect to $n$, then due to (4.38) we get

$$
\begin{align*}
\frac{1}{2} \sum_{j=1}^{n} |f_{n+1} - f_n - |f_{n+1} - f_n| | &\leq F_+(n+1,n+1,x_{n+1}) - \varepsilon \\
v_{n+1} - v_n &\leq -F_-(n+1,n+1,x_{n+1}) - \varepsilon.
\end{align*}
$$

(4.39)

Similarly, because the sum (4.37) is nondecreasing with respect to $n$ we obtain for the difference $v_{n+1} - v_n$ the estimate

$$
\begin{align*}
\frac{1}{2} \sum_{j=1}^{n} |f_{n+1} - f_n + |f_{n+1} - f_n| | &\leq F_+(n+1,n+1,x_{n+1}) - \varepsilon \\
v_{n+1} - v_n &\geq -F_-(n+1,n+1,u_{n+1} - v_{n+1}) - \varepsilon.
\end{align*}
$$

(4.40)

As a result we obtain $u_1 < 0$, $v_1 < 0$ and both inequalities (4.39), (4.40) are valid. Now we have three possibilities. (1) If $u_n < 0$, $v_n < 0$ for all $n \geq 1$ then by virtue of (4.38) and conditions

$$
F_+(n,j,x_j) \geq 0, \quad F_-(n,j,x_j) \leq 0,
$$

we obtain

$$
\begin{align*}
\frac{1}{2} \left( f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) - \varepsilon n \leq u_n \leq u_n - v_n = x_n \\
\leq - v_n \leq \frac{1}{2} \left( f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) + \varepsilon n.
\end{align*}
$$

From here and arbitrariness of $\varepsilon > 0$ we derive the estimates (4.33).

Now consider two cases when inequalities $u_n < 0$ and $v_n < 0$ are not valid for all $n \geq 1$. (2) Assume that there exists first moment $n_0 > 1$ such that:

$$
\begin{align*}
u_n < 0, &\quad 1 \leq n \leq n_0, \quad u_{n_0+1} \geq 0, \quad v_n \leq 0, \quad 1 \leq n \leq n_0,
\end{align*}
$$


but \( u_{n_{0} + 1} - v_{n_{0} + 1} \geq 0 \). Then we have \( u_{n_{0} + 1} - u_{n_{0}} \geq 0 \). From the other side by virtue of (4.34) and (4.39):

\[
u_{n_{0} + 1} - u_{n_{0}} \leq F_{+}(n_{0} + 1, n_{0} + 1, u_{n_{0} + 1} - v_{n_{0} + 1}) - \varepsilon = -\varepsilon < 0.
\]

(3) Consider now the second possible case when there exists first moment \( n_{0} > 1 \) such that:

\[
v_{n} < 0, \quad 1 \leq n \leq n_{0}, \quad v_{n_{0} + 1} \geq 0, \quad u_{n} \leq 0, \quad 1 \leq n \leq n_{0},
\]

and besides \( v_{n_{0} + 1} - u_{n_{0} + 1} \geq 0 \). Then we have \( v_{n_{0} + 1} - v_{n_{0}} \geq 0 \). From the other side due to (4.34) and (4.40):

\[
u_{n_{0} + 1} - v_{n_{0}} \leq - F_{-}(n_{0} + 1, n_{0} + 1, u_{n_{0} + 1} - v_{n_{0} + 1}) - \varepsilon = -\varepsilon.
\]

Contradictions obtained in the possibilities (2) and (3) prove the estimates (4.33), Theorem 4.6 is proven.

Remark 4.1. The estimates of the solutions of linear scalar Volterra equation:

\[
x_{n} = \sum_{j=1}^{n} K(n,j)x_{j} + f_{n}, \quad n \geq 1.
\]

where the function \( K(n,j) \) be nonpositive and nondecreasing with respect to \( n, \ n \geq j \) and perturbations satisfy condition (4.32) of Theorem 4.6 (with \( f_{0} = 0 \)).

\[
\sum_{j=1}^{n} (f_{j} - f_{j-1})_{-} \leq x_{n} \leq \sum_{j=1}^{n} (f_{j} - f_{j-1})_{+}.
\]

Here

\[
f_{+} = \max(0, f) = \frac{1}{2} (|f| + f), \quad f_{-} = \min(0, f) = f - f_{+}.
\]

Example 4.4. Consider the following equations:

\[
x_{n+1} - x_{n} = -a_{0}x_{n} - a_{1}x_{n-1}.
\]

Necessary and sufficient conditions for the boundedness of solutions are (See [5, pp. 435]):

\[
a_{0} + a_{1} \geq 0, \quad -a_{0} + a_{1} + 2 \geq 0, \quad a_{0} + 3a_{1} - 2 \leq 0,
\]

where coefficients \( a_{0} \) and \( a_{1} \) are constants.

4.6. Boundedness of certain Volterra systems with dissipative nonlinearity

The conditions of boundedness of the process are quite important in applications. For general discrete-time Volterra systems, the boundedness conditions are formulated in [1] through Lyapunov’s second method. But, owing to the difficulties in constructing the Lyapunov functions, concrete systems may be studied by methods that take into account the specifics of these systems.
In this paper, we study the boundedness conditions for scalar nonlinear systems:

\[ \Delta x_n = x_{n+1} - x_n = b_n - \sum_{j=0}^{n} a_j f(x_{n-j}), \quad n \geq 0. \]  

(4.41)

Here the discrete time \( n \in N := \{0, 1, 2, \ldots\} \), the functions \( f : \mathbb{R} \to \mathbb{R} \), and number sequences \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \), are given. The solution of equation (4.41) is determined for a given initial condition \( x_0, n = 0 \), and is sometimes denoted by \( x(n, x_0) \).

System (4.41) is said to be bounded if \( \sup_{n \geq 0} \{|x(n, x_0)|\} < \infty \) for any finite \( x_0 \).

**Theorem 4.7.** System (4.41) is bounded under the following conditions:

\[ a_n > 0, \quad a = \sum_{n=0}^{\infty} a_n < \infty; \]  

(4.42)

\[ b_n \to 0, \quad n \to \infty, \quad b = \sup_{n \geq 0} \{|b_n|\} < \infty; \]  

(4.43)

\[ \lim_{x \to \infty} f(x) > 0, \quad \lim_{x \to -\infty} f(x) < 0; \]  

\[ f(x) \geq -r, \quad x \in \mathbb{R}; \quad |f(x)| < \infty, \quad |x| < \infty; \]  

(4.44)

where \( r \) is a given positive constant, and \( \limsup \) and \( \liminf \) denote the upper and lower limit, respectively.

**Proof.** First we prove that

\[ \sup_{n \geq 0} \{x_n\} < \infty. \]  

(4.45)

Let us assume that inequality (4.45) is not satisfied. Note that, by virtue of (4.42)–(4.44), the solution \( x(n, x_0) \) is finite for any finite \( x_0 \) and \( n \in N \). Hence, by our assumption

\[ \lim_{n \to \infty} x_n = \infty. \]  

(4.46)

Let \( c = b + ra \). Then, by virtue of (4.41):

\[ |\Delta x_n| \leq c. \]  

(4.47)

Let us introduce a sequence of levels \( \{y_n\}_{n \geq 0} \), where \( y_n = x_0 + 2nc, \) \( n \geq 1 \), and a sequence of instants \( \{\tau_n\}_{n \geq 0} \) and \( \{s_n\}_{n \geq 0} \). The instant \( \tau_n \) is the first instant if \( x(\tau_n, x_0) \geq y_n \). The instant \( s_n, n \geq 1 \) is defined by the relations:

\[ x(s_n - 1) \leq y_{n-1}, \quad x(s_n) \geq y_{n-1}; \]

\[ y_{n-1} < x(j, x_0) \leq x(\tau_n), \quad s_n < j \leq \tau_n. \]  

(4.48)

In other words, \( s_n \) is the first instant when the solution attains (or exceeds) the level \( y_{n-1} \). Furthermore, the level \( y_n \) is attained (or exceeded) earlier than the solution becomes once again less than or equal to \( y_{n-1} \). By virtue of (4.47), (4.48), and the definition of \( y_n \), we find that \( s_n < \tau_n \), where \( s_n \to \infty \) and \( \tau_n \to \infty \) as \( n \to \infty \).
Moreover, among the instants $s_n, s_{n+1}, \ldots, s_{n}$, there certainly exists an instant $t_n$, $t_n \to \infty$ as $n \to \infty$, at which:
\[
\Delta x(t_n, x_0) = x(t_n + 1, x_0) - x(t_n, x_0) \geq 0,
\]
\[
t_n \to \infty, \quad n \to \infty.
\] (4.49)

Let
\[
T_n = \min \left[ t_n, \frac{1}{2c} x(t_n, x_0) \right].
\] (4.50)

By (4.46), (4.50) and (4.51), we find that $T_n \to \infty$ as $n \to \infty$. Furthermore, at any instant $j$, $t_n - T_n \leq j \leq t_n$, by virtue of (4.47) and (4.51):
\[
x(t_n, x_0) - x_j = \sum_{l=j}^{t_n-1} \Delta x_l \leq c(t_n - j) \leq \frac{1}{2} x(t_n, x_0).
\] (4.51)

Consequently
\[
x_j \geq \frac{1}{2} x(t_n, x_0), \quad t_n - T_n \leq j \leq t_n.
\] (4.52)

Since $t_n \to \infty$ as $n \to \infty$, the solution $x(t_n, x_0)$, by virtue of (4.46), tends to infinity as $n$ tends to infinity. Consequently, by (4.44), there exists an instant $n_0$ such that $f(x(t_n, x_0)) \geq c_1 > 0$ for all $n \geq n_0$. Hence, (4.44) and (4.52) imply that
\[
f(x_j) \geq c_1 > 0, \quad n \geq n_0, \quad t_n - T_n \leq j \leq t_n,
\] (4.53)

for all $n \geq n_0$. Moreover, for $n \geq n_0$ we have:
\[
\Delta x(t_n, x_0) = b(t_n) - \sum_{j=0}^{T_n} a_j f(x(t_n - j, x_0)) - \sum_{j=T_n+1}^{t_n} a_j f(x(t_n - j, x_0)),
\] (4.54)

where the last sum on the right side is zero if $T_n = t_n$. By virtue of (4.44), (4.53), and (4.54), we obtain
\[
\Delta x(t_n, x_0) \leq b(t_n) - c_1 \sum_{j=0}^{T_n} a_j + r \sum_{j=T_n+1}^{t_n} a_j.
\]

The right hand side of the last inequality is negative for all sufficiently large $n$, because, by virtue of (4.42)−(4.44):
\[
b(t_n) \to 0, \quad \sum_{j=T_n+1}^{t_n} a_j \to 0, \quad c_1 \sum_{j=0}^{T_n} a_j \to c_1 a > 0,
\]
as $n \to \infty$. But the negativity of $\Delta x(t_n, x_0)$ contradicts definition (4.49) of the instant $t_n$. Hence, relation (4.45) holds.

**Remark 4.2.** The proof of inequality (4.45) shows that, instead of the condition $f(x) \geq -r$, it suffices to assume that the inequality
\[
\inf_{n \geq 0} \{ f(x(n, x_0)) \} > -\infty, \quad n \in N,
\] (4.55)

is satisfied along any solution $x(n, x_0)$.
Now to prove Theorem 4.7, it suffices to show that
\[
\inf_{n \geq 0} \{x(n,x_0)\} > -\infty, \quad n \in N. \quad (4.56)
\]

Note that, by virtue of (4.44) and (4.45), for any solution \(x(n,x_0)\), we have
\[
\sup_{n \geq 0} \{|f(x(n,x_0))|\} < \infty. \quad (4.57)
\]

Introducing a process \(y_n = -x_n\) and a function \(F(y) = -f(-y)\), we find that
\[
y_{n+1} - y_n = -(x_{n+1} - x_n) = -b_n + \sum_{j=0}^{n} a_j f(-y_{n-j})
= -b_n - \sum_{j=0}^{n} a_j F(y_{n-j}), \quad n \in N, \quad y_0 = -x_0. \quad (4.58)
\]

The function \(F(y)\) on the right side has the following properties similar to (4.44):
\[
\lim_{y \to \infty} F(y) = -\lim_{y \to -\infty} f(-y) = -\lim_{y \to \infty} f(y) > 0,
\]
\[
\lim_{y \to -\infty} F(y) = -\lim_{y \to -\infty} f(-y) = -\lim_{y \to \infty} f(y) > 0,
\]
\[
|F(y)| < \infty, \quad |y| < \infty. \quad (4.59)
\]

Finally, by Remark 4.2, it remains to varify, by virtue of (4.55), that the inequality
\[
\inf_{n \geq 0} \{F(y(n,x_0))\} > -\infty, \quad n \in N,
\]
holds for any solution \(y(n,x_0)\). But this inequality follows directly from (4.57) and the definitions of \(y_n\) and \(F(y)\). Comparing (4.41) and (4.58), we find that the function \(y(n,x_0)\) satisfies an equation of the same type as (4.41). But estimate (4.45) holds for the solutions of Eq. (4.41). Therefore:
\[
\sup_{n \geq 0} \{y(n,x_0)\} < \infty.
\]

From this inequality and the definition of \(y_n\) we obtain (4.45). This completes the proof of Theorem 4.7.

4.7. Systems with sign definite nonlinearity

The following theorems 4.8–4.10 are formulated without proof (see Automation and Remote Control, Vol. 60, No. 3, 1999).
Theorem 4.8. Let there exist an \( n \geq 0 \) such that \( a_j > 0 \) for \( j \geq n \), and let the conditions:

\[
q = \sum_{i=0}^{\infty} \left| \sum_{j=i+1}^{\infty} a_j \right| < \infty, \quad A_i \geq 0, \quad n \geq 0, \quad A_\infty > 0;
\]

\[
f(x) \geq -\lambda, \quad x f(x) > 0, \quad x \neq 0,
\]

be satisfied, where \( \lambda \) is a positive constant. Then system (4.41) is bounded.

Theorem 4.9. Let there exist and \( n \geq 0 \) such that all \( a_j < 0 \) for \( j \geq n \), and \( a_i > 0 \) for \( i < n \). Furthermore, let conditions of Theorem 4.7 and the inequalities:

\[
A_i \geq A_\infty, \quad i \geq n,
\]

be satisfied. Then system (4.41) is bounded.

Theorem 4.10. Assume that

\[
b_n \to 0, \quad n \to \infty, \quad \sum_{j=0}^{\infty} |a_j| < \infty;
\]

\[
q_1 = \sum_{j=0}^{\infty} j^2 (-a_{j+1} + a_j) < \infty, \quad -a_{j+1} + a_j \geq 0, \quad j \geq 0;
\]

\[
q_2 = \sum_{j=0}^{\infty} j^2 (a_{j+1} - 2a_j + a_{j-1}) < \infty, \quad a_{j+1} - 2a_j + a_{j-1} \geq 0, \quad j \geq 0,
\]

where \( a_{-1} \in [0, 2) \) is a constant.

The continuous function \( f(x) \) satisfies the conditions:

\[
x f(x) > 0, \quad x \neq 0, \quad |f(x)| \leq |x|.
\]

Then every bounded solution \( x_n \) of Eq. (4.41) satisfies the equality \( \lim_{n \to \infty} x_n = 0 \).

In [59–61], the propagation of perturbations in viscoelastic media is described through a Volterra system containing the space variable as a parameter in the nonlinear term. Owing to the difficulties encountered in analysing this system, it is more convenient to approximate the system by a Volterra difference equation of the form:

\[
x_n = b_n - \sum_{j=0}^{n} a_j f(x_{n-j}), \quad n \geq 0.
\] (4.60)

In establishing the properties of the initial system, a key role is played by the boundedness conditions of system (4.60), which are not dependent on the concrete type of nonlinearity \( f \) or, consequently, on the space variable.

Moreover, Eq. (4.60) is also used in implicit numerical schemes. Let the function \( f \) be continuous and let

\[
a_n \geq 0, \quad a_{n+j} - a_n \leq 0, \quad n, \quad j \geq 0, \quad x f(x) \geq 0.
\]
Then, for all \( n \geq 0 \):
\[
|x_n| \leq \sum_{j=0}^{\infty} |b_{j+1} - b_j| + \sup_{j \geq 0} |b_j| + |b_0|.
\] (4.61)

Note that inequality (4.61) does not depend on the concrete form of the function \( f \).

To demonstrate (4.61), let us examine a few possible cases.

First we assume that \( x_0 \geq 0 \). Let \( s_0 \) be the instant when the solution \( x_n \) passes through zero for the first time, i.e., \( x_n \geq 0 \) for \( 0 \leq n \leq s_0 \) and \( x_{s_0+1} < 0 \). Let \( \tau_0 \) be the first instant when the solution passes through zero after the instant \( s_0 \), i.e., \( x_n \leq 0 \) for \( s_0 + 1 \leq n \leq \tau_0 \) and \( x_{\tau_0+1} > 0 \). The instants \( s_n \) and \( \tau_n \), \( n \geq 1 \), successive passages of the solution through zero are defined along similar lines. At the instants \( s_n \) the solution passes from nonnegative values to negative values, and at the instants \( \tau_n \) the solution passes from nonpositive values to positive values.

Introducing two sequences \( \{\alpha_n\}_{n \geq 0} \) and \( \{\beta_n\}_{n \geq 0} \) of nonnegative quantities:
\[
\alpha_n = \sum_{j=0}^{n} a_j \max[f(x_{n-j}), 0] \geq 0,
\]
\[
\beta_n = -\sum_{j=0}^{n} a_j \min[f(x_{n-j}), 0] \geq 0,
\]
we can express Eq. (4.60) as
\[
x_n = b_n - \alpha_n + \beta_n.
\] (4.62)

Therefore, for any \( n \geq 0 \), we have
\[
x_n \leq b_n + \beta_n.
\] (4.63)

Taking an arbitrary instant \( \tau \geq 1 \) at which \( x_\tau > 0 \), let us estimate \( x_\tau \). By virtue of (4.63):
\[
x_\tau \leq b_\tau + \beta_\tau.
\] (4.64)

In this case, \( \beta_0 = 0 \). Therefore, it suffices to restrict the study to the case of \( \tau \geq 1 \). We have
\[
\beta_\tau = \sum_{j=0}^{\tau-1} \Delta \beta_j, \quad \Delta \beta_j = \beta_{j+1} - \beta_j.
\]

Let \( Z \) denote the set of all instants between 0 and \( \tau - 1 \) that belong to one of the intervals \([s_n, \tau_n - 1]\), \( n \geq 0 \). Let \( k \leq \tau - 1 \) be an arbitrary instant that does not belong to \( Z \). Then \( x_{k+1} \geq 0 \) at the instant \( k + 1 \). Therefore
\[
\Delta \beta_k = \beta_{k+1} - \beta_k = -\sum_{j=0}^{k+1} a_{k+1-j} \min[f(x_j), 0] + \sum_{j=0}^{k} a_{k-j} \min[f(x_j), 0]
\]
\[
= \sum_{j=0}^{k} [a_{k-j} - a_{k+1-j}] \min[f(x_j), 0] \leq 0.
\]
Consequently
\[ \beta_\tau \leq \sum_{j \in Z} \Delta \beta_j. \] (4.65)

But, by virtue of (4.62):
\[ \Delta \beta_j = \Delta x_j + \Delta x_j - \Delta b_j. \] (4.66)

Substituting (4.66) into (4.65), let us examine individual terms. Since \( x(\tau_j) - x(s_j) \leq 0 \) for any \( j \), we have
\[ \sum_{j \in Z} \Delta x_j = x(\tau_m) - x(s_m) + x(\tau_n - 1) \]
\[ - x(s_n - 1) + \cdots + x(\tau_0) - x(s_0) \leq 0. \] (4.67)

Here \( \tau_m = \max\{\tau_n: \tau_n < \tau\} \). Moreover
\[ x(\tau_j) - x(s_j) = \sum_{k=0}^{\tau_j - 1} a_{\tau_j - k} \max[f(x_k), 0] - \sum_{k=0}^{\tau_j - 1} a_{s_j - k} \max[f(x_k), 0] \]
\[ = \sum_{k=0}^{s_j} [a_{\tau_j - k} - a_{s_j - k}] \max[f(x_k), 0] \leq 0. \]

Therefore
\[ \sum_{j \in Z} \Delta x_j = \sum_{j=0}^{m} [x(\tau_j) - x(s_j)] \leq 0. \] (4.68)

Thus, by virtue of (4.64)–(4.68), \( x_\tau \leq b_\tau + \sum_{j \in Z} |\Delta b_j| \) for \( x_\tau > 0 \).

Let us now estimate \( x_\tau \) under the assumption that \( x_0 < 0 \) and \( x_\tau > 0 \). In this case, the sequence \( \{s_n\}_{n \geq 1} \) begins with the element \( s_1 \) and the sequence \( \{\tau_n\}_{n \geq 0} \) begins with the element \( \tau_0 \), i.e., \( x_n \leq 0 \) for \( 0 \leq n \leq \tau_0, x_{\tau_0 + 1} > 0, x_n > 0 \) for \( \tau_0 + 1 \leq n \leq s_1, \) and \( x_{s_1 + 1} < 0 \).

The remaining terms of these sequences are defined along similar lines.

Then, as before, estimate (4.64) holds. Moreover, as for (4.65), we have
\[ \beta_\tau = \sum_{j \in Z} \Delta \beta_j + \beta_0 \leq \sum_{j=1}^{m} (\beta_{\tau_j - 1} - \beta_{s_j}) + \beta_{\tau_0}. \] (4.69)

The sum on the right hand side of (4.69), by virtue of (4.66)–(4.68), satisfies the estimate
\[ \sum_{j=1}^{m} (\beta_{\tau_j - 1} - \beta_{s_j}) \leq \sum_{j \in Z_1} |\Delta b_j|, \] (4.70)

where \( Z_1 \) is the set of all instants between 0 and \( \tau - 1 \) that belong to one of the intervals \( [s_n, \tau_n - 1], n \geq 1 \). Now, by virtue of (4.62):
\[ \beta_{\tau_0} = \sum_{j=0}^{\tau_0 - 1} \Delta \beta_j + \beta_0 = \sum_{j=0}^{\tau_0 - 1} \Delta [x_j - b_j + \beta_0]. \] (4.71)
By the definitions of the instants $\tau_0$ and sequence $\{x_n\}_{n \geq 0}$, we have

$$\sum_{j=0}^{\tau_0-1} \Delta x_j \leq -x_0, \quad \sum_{j=0}^{\tau_0-1} \alpha_j = 0.$$  \hspace{1cm} (4.72)

Thus, by virtue of (4.64) and (4.69)–(4.72), we find that

$$x_\tau \leq \sum_{j=0}^{\infty} |\Delta b_j| + |b_0| + |b_\tau|.$$  \hspace{1cm} (4.73)

To demonstrate estimate (4.61) for $x_\tau < 0$, let us introduce the new variable $y_n = -x_n$ and the function $g(x) = -f(-x)$. By virtue of (4.60), we have

$$y_n = -b_n - \sum_{j=0}^{n} a_j g(y_{n-j}).$$

Since $xg(x) \geq 0$, this equation for $y_n$ is of the same type as Eq. (4.60). Hence, an estimate of the form (4.73) holds for $y_\tau$ at the points $\tau$ at which $x_\tau < 0$ (i.e. $y_\tau > 0$). Hence $x_n$ satisfies (4.61) for all $n$.

4.8. Linear equation under unknown perturbations

Let us consider the scalar system:

$$x_{n+1} = -\sum_{i=0}^{n} a_{n-i} x_i - b_n, \quad n \geq 0,$$  \hspace{1cm} (4.74)

where $a_n$ is the given sequence of coefficients and $b_n$ are the system perturbations. The initial position $x_0$ of the system is given for $n = 0$. To study the properties of system (4.74), we define on its solutions the functional $V_0 = V_0(n,x_0,\ldots,x_n)$ equal to

$$V_0 = (2 - a_{-1})x_n^2 + a_{n+1}\left(\sum_{i=0}^{n} x_i\right)^2 - \sum_{i=0}^{n} (a_{n+1-i} - a_{n-i})\left(\sum_{j=i}^{n} x_j\right)^2,$$  \hspace{1cm} (4.75)

where $a_{-1}$ is a constant which is chosen below.

Using functional from (3.12) and its first difference (3.13) we obtain that

$$\Delta V_0 = J_0(n,x) - 2b_n x_{n+1},$$

$$J_0(n,x) = (a_{n+2} - a_{n+1})\left(\sum_{i=0}^{n+1} x_i\right)^2 - \sum_{i=0}^{n+1} (a_{n+2-i} - 2a_{n+1-i} + a_{n-i})\left(\sum_{j=i}^{n+1} x_j\right)^2 - (2 - a_{-1})x_n^2.$$  \hspace{1cm} (4.76)

The relationships (4.74)–(4.76) allow us to draw some conclusions about the boundedness of the solutions of Eq. (4.74).
Theorem 4.11. Let there be a constant $a_{-1}$ such that the inequalities:

$$a_{-1} < 2, \quad a_j \geq 0, \quad a_{j+1} \leq a_j,$$

$$a_{j+2} - 2a_{j+1} + a_j \geq 0, \quad j \geq 0,$$

are valid; and also let

$$\sum_{j=0}^{\infty} a_j < \infty. \quad (4.77)$$

Then, any solution of Eq. (4.74) is bounded if

$$\sup_{n \geq 0} \{|b_n|\} < \infty. \quad (4.79)$$

Proof. On the basis of the second Lyapunov method for the Volterra equation, if conditions (4.77) are satisfied, then the system:

$$x_{n+1} = -\sum_{i=0}^{n} a_{n-i}x_i, \quad n \geq 0; \quad (4.80)$$

is uniformly asymptotically stable at the initial instant because functional (4.75) is positive definite and, with regard for (4.76), its first-order difference is negative definite along the trajectories of system (4.80).

We denote by $R_j$ the resolvent of system (4.80) which obeys the relationships:

$$R_{n+1} = -\sum_{i=0}^{n} a_{n-i}R_i, \quad n \geq 0; \quad R_0 = I. \quad (4.81)$$

The resolvent satisfies the inequality:

$$\sum_{j=0}^{\infty} |R_j| < \infty, \quad (4.82)$$

by virtue of uniform asymptotic stability of system (4.80) and condition (4.78). Additionally, Eq. (4.74) has the following solution:

$$x_n = R_n x_0 + \sum_{i=0}^{n-1} R_{n-i}b_i, \quad n \geq 0, \quad (4.83)$$

where the sum in the right hand side is assumed to be zero for $n = 0$.

Hence, it follows from (4.79) and (4.83) that

$$|x_n| \leq |R_n x_0| + \sup_{n \geq 0} \{|b_n|\} \sum_{i=0}^{\infty} |R_i|, \quad (4.84)$$

which, with regard for (4.82), proves that the solution $x_n$ is bounded under conditions (4.77)–(4.79). Theorem 4.11 is proven. $\square$
4.9. Two-dimensional example

Example 4.5. Let us consider for \( n > 0 \) two dimensional perturbed Volterra difference equations:

\[
x_{n+1} = x_0 \cos\left(\frac{\pi}{2}(n + 1)\right) + y_0 \sin\left(\frac{\pi}{2}(n + 1)\right) - 2 \sum_{j=0}^{n} \frac{x_j}{j+1} \sin\left(\frac{\pi}{2}(n - j)\right),
\]

\[
y_{n+1} = -x_0 \sin\left(\frac{\pi}{2}(n + 1)\right) + y_0 \cos\left(\frac{\pi}{2}(n + 1)\right) - 2 \sum_{j=0}^{n} \frac{x_j}{j+1} \cos\left(\frac{\pi}{2}(n - j)\right),
\]

(4.85)

where \( x_0 \) and \( y_0 \) are prescribed values.

The equations without perturbations corresponding to (4.85) have the form

\[
x_{n+1} = x_0 \cos\left(\frac{\pi}{2}(n + 1)\right) + y_0 \sin\left(\frac{\pi}{2}(n + 1)\right),
\]

\[
y_{n+1} = -x_0 \sin\left(\frac{\pi}{2}(n + 1)\right) + y_0 \cos\left(\frac{\pi}{2}(n + 1)\right), \quad n \geq 0.
\]

(4.86)

From Eqs. (4.86) it follows that the solutions of the nonperturbed system (4.86) are uniformly bounded for all \( n \geq 0 \).

Hence

\[
x_n = \left[2 \sum_{j=0}^{n-1} \cos^2\left(\frac{\pi}{2}j\right) + 1\right] \cos\left(\frac{\pi}{2}n\right) x_0
\]

\[
+ \left[2 \sum_{j=0}^{n-1} \sin^2\left(\frac{\pi}{2}j\right) + 1\right] \sin\left(\frac{\pi}{2}n\right) y_0
\]

\[
= (n + 1) \left[x_0 \cos\left(\frac{\pi}{2}n\right) + y_0 \sin\left(\frac{\pi}{2}n\right)\right],
\]

and for the component \( y_n \) we get

\[
y_n = (n + 2) \left[-x_0 \sin\left(\frac{\pi}{2}n\right) + y_0 \cos\left(\frac{\pi}{2}n\right)\right].
\]

If \( x_0 \neq 0 \) or \( y_0 \neq 0 \) then the solutions of perturbed equations in (4.85) are unbounded.
Now we can show that at the same time the solutions of perturbed equations in (4.85) are bounded in average. For this let us use equality

\[
\begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1}
\begin{bmatrix}
\sin(z) + \sin(2z) + \cdots + \sin(nz)
\end{bmatrix}
\begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}
= \begin{bmatrix}
\cos(z/2) - \cos(3z/2) + \cdots + \cos((2n-1)z/2)
\end{bmatrix}
\begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1}.
\]

It means that

\[
\sum_{j=1}^{n} \sin(z_j) = \begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1} \begin{bmatrix}
\cos \left( \frac{z}{2} \right) - \cos \left( \frac{2n+1}{2} \right)
\end{bmatrix}.
\]

Let us summarize the sum

\[
\sum_{j=1}^{n} j \sin(z_j) = \sum_{j=1}^{n} \sin(z_j) + \sum_{j=2}^{n} \sin(z_j) + \cdots + \sin(nz)
= \begin{bmatrix}
\cos(z/2) - \cos(2z/2) + \cos(z/2)
\end{bmatrix}
\begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1}
= \begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1} \sum_{j=1}^{n} \cos \left( \frac{2j-1}{2} \right) - n \cos \left( \frac{2n+1}{2} \right).
\]

Further

\[
\sum_{j=1}^{n} \cos \left( \frac{2j-1}{2} \right) = \begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1}
\begin{bmatrix}
\cos(z/2) + \cos(z/2) + \cdots \\
+ \cos(z(2n-1)/2)
\end{bmatrix}
\begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}
= \begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1} \begin{bmatrix}
\sin(z) + \sin(3z) - \sin(z) \\
+ \sin(3z) - \sin(2z) + \cdots \\
+ \sin(nz) - \sin(z(n-1))
\end{bmatrix}
= \begin{bmatrix}
2 \sin \left( \frac{z}{2} \right)
\end{bmatrix}^{-1} \sin(zn).
Hence
\[
\sum_{j=1}^{n} j \sin(xj) = \left[4 \sin^2 \left(\frac{x}{2}\right)\right]^{-1} \sin(zn)
- \left[2 \sin \left(\frac{x}{2}\right)\right]^{-1} n \cos \left(\frac{2n + 1}{2}\right).
\]

Now using the equality of \([\cos(x) + \cdots + \cos(zn)]\):
\[
\sum_{j=1}^{n} \cos(xj) = \left[2 \sin \left(\frac{x}{2}\right)\right]^{-1} \left[\sin \left(\frac{2n + 1}{2}\right) - \sin \left(\frac{x}{2}\right)\right],
\]

and summarizing
\[
\sum_{j=1}^{n} j \cos(xj) = \left[2 \sin \left(\frac{x}{2}\right)\right]^{-1} \left[n \sin \left(\frac{2n + 1}{2}\right) - \sum_{j=1}^{n} \sin \left(\frac{2j - 1}{2}\right)\right].
\]

But
\[
\sum_{j=1}^{n} \sin \left(\frac{2j - 1}{2}\right) = \left[2 \sin \left(\frac{x}{2}\right)\right]^{-1} \times \left[\sin(\frac{x}{2}) + \sin(\frac{x}{2}) + \cdots + \sin(x(2n - 1)/2)\right]
\times \left[2 \sin \left(\frac{x}{2}\right)\right] = \left[2 \sin \left(\frac{x}{2}\right)\right]^{-1} \left[\cos(0) - \cos(x) + \cos(x) + \cdots + \cos(x(n - 1)) - \cos(zn)\right] = \left[2 \sin \left(\frac{x}{2}\right)\right]^{-1} \left[1 - \cos(zn)\right].
\]

It means that the first component \(x_n\) of perturbed equation (4.85) can be represented in the form
\[
x_n = x_0 \cos \left(\frac{\pi}{2} n\right) + y_0 \sin \left(\frac{\pi}{2} n\right) + y_0 \left[4 \sin^2 \left(\frac{\pi}{4}\right)\right]^{-1} \sin \left(\frac{\pi}{2} n\right)
- \left[2 \sin \left(\frac{\pi}{4}\right)\right]^{-1} y_0 n \cos \left(\frac{\pi}{4} (2n + 1)\right)
+ \left[2 \sin \left(\frac{\pi}{4}\right)\right]^{-1} x_0 n \sin \left(\frac{\pi}{4} (2n + 1)\right)
- \left[4 \sin^2 \left(\frac{\pi}{4}\right)\right]^{-1} x_0 \left[1 - \cos \left(\frac{\pi}{2} n\right)\right].
\]

(4.87)
Taking the modulus of both parts of Eq. (4.87) and summarizing from 0 to $N$:

\[ \sum_{j=0}^{N} |x_j| \leq |x_0| \sum_{j=0}^{N} \cos \left( \frac{\pi}{2} j \right) + \frac{3}{2} |y_0| \sum_{j=0}^{N} \sin \left( \frac{\pi}{2} j \right) \]

\[ + \frac{1}{\sqrt{2}} |y_0| \sum_{j=0}^{N} j \cos \left( \frac{\pi}{4} (2j + 1) \right) + \frac{1}{\sqrt{2}} |x_0| \sum_{j=0}^{N} j \sin \left( \frac{\pi}{4} (2j + 1) \right) \]

\[ + \frac{1}{2} |x_0| \sum_{j=0}^{N} \left[ 1 - \cos \left( \frac{\pi}{2} j \right) \right] \]

Further let us divide both parts of this inequality on $N$ and take upper limit as $N \to \infty$. Then

\[ \lim_{N \to \infty} \frac{1}{N} \left[ |x_0| \sum_{j=0}^{N} \cos \left( \frac{\pi}{2} j \right) + \frac{3}{2} |y_0| \sum_{j=0}^{N} \sin \left( \frac{\pi}{2} j \right) \right] = 0, \]

and we obtain

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} |x_j| \leq \lim_{N \to \infty} \frac{1}{N} \left[ \frac{1}{\sqrt{2}} |y_0| \sum_{j=0}^{N} j \cos \left( \frac{\pi}{4} (2j + 1) \right) \right] \]

\[ + \lim_{N \to \infty} \frac{1}{N} \left[ \frac{1}{\sqrt{2}} |x_0| \sum_{j=0}^{N} j \sin \left( \frac{\pi}{4} (2j + 1) \right) \right] \]

\[ + \lim_{N \to \infty} \frac{1}{N} \left[ \frac{1}{2} |x_0| \sum_{j=0}^{N} \left[ 1 - \cos \left( \frac{\pi}{2} j \right) \right] \right]. \]

Let us estimate the first sum

\[ \sum_{j=0}^{N} j \cos \left( \frac{\pi}{4} (2j + 1) \right) = \frac{\sqrt{2}}{2} \sum_{j=0}^{N} j \left[ \cos \left( \frac{\pi}{2} j \right) - \sin \left( \frac{\pi}{2} j \right) \right] \]

\[ = \frac{\sqrt{2}}{2} \left[ 2 \sin \left( \frac{\pi}{4} \right) \right]^{-1} N \sin \left( \frac{\pi}{2} \frac{2N+1}{2} \right) - \frac{\sqrt{2}}{2} \left[ 2 \sin \left( \frac{\pi}{4} \right) \right]^{-2} \left[ 1 - \cos \left( \frac{\pi}{2} N \right) \right] \]

\[ - \frac{\sqrt{2}}{2} \left[ 2 \sin \left( \frac{\pi}{4} \right) \right]^{-2} \sin \left( \frac{\pi}{2} N \right) + \frac{\sqrt{2}}{2} \left[ 2 \sin \left( \frac{\pi}{4} \right) \right]^{-1} N \cos \left( \frac{\pi}{2} \frac{2N+1}{2} \right). \]
The second sum will be estimated similarly:
\[
\sum_{j=0}^{N} j \sin\left(\frac{\pi}{2} j + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \sum_{j=0}^{N} j \left[ \sin\left(\frac{\pi}{2} j\right) + \cos\left(\frac{\pi}{2} j\right) \right] = \sqrt{2} \left[ [2 \sin(\frac{\pi}{4})]^{-2} \sin(\frac{\pi}{4}N) \right. \\
- [2 \sin(\frac{\pi}{4})]^{-1} N \cos(\frac{\pi}{4}N + \frac{1}{2}) \\
+ [2 \sin(\frac{\pi}{4})]^{-1} N \sin(\frac{\pi}{4}N + \frac{1}{2}) \\
- [2 \sin(\frac{\pi}{4})]^{-2} [1 - \cos(\frac{\pi}{4}N)] \right].
\] \tag{4.90}

The third sum
\[
\sum_{j=0}^{N} \left[ 1 - \cos\left(\frac{\pi}{2} j\right) \right],
\]
represents nondecreasing sequence as \(N \to \infty\) and moreover
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \left[ 1 - \cos\left(\frac{\pi}{2} j\right) \right] = 1.
\]

Taking into account relations from (4.89) and (4.90) we have
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} |x_j| \leq \frac{3}{4} |x_0| + \frac{1}{4} |y_0|.
\]

Similar arguments show that upper limit of
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} |y_j|,
\]
will be also bounded in average.

As a result we have shown that the solutions of unperturbed equations are bounded, the solutions of perturbed equations are unbounded, and the solutions of perturbed equations are bounded in average.

References


