Mixed $\mathcal{L}_1/\mathcal{H}_2/\mathcal{H}_\infty$ Control Design: Numerical Optimization Approaches

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Abstract

This paper presents some new approaches to mixed performance control problems of linear systems. The design techniques proposed in this paper are based on numerical search of the norm bounded stable transfer matrix $Q$ in the $H_\infty$ and $H_2$ suboptimal controller parameterizations so that the additional performance specifications are satisfied. The design problems are then converted to some finite dimensional nonlinear unconstrained optimization problems by explicitly parameterizing the $H_\infty$ or $H_2$ norm bounded stable transfer matrix $Q$ for any fixed order. Finally, some two-stage optimization algorithms are applied to find the optimal parameters. Numerical examples have shown significant performance improvements of the proposed algorithms over those in the existing literature.

1 Introduction

The fundamental objective of a feedback control system design is to achieve desired performance despite of model uncertainties and external disturbances. It is well-known in the control community that there are intrinsic conflicts between achievable performance and system robustness. A well thought control system design is to make some suitable tradeoffs between performance and system robustness. It is therefore desirable to develop design techniques that can optimally and systematically perform such performance and robustness tradeoffs. It is therefore not surprising that multiobjective (or mixed performance) optimal control has become a crucially important research area in the last decade or so.

Many approaches have been proposed in the literature to solve mixed performance problems. An overview of various approaches to multi-objective design is presented by Vroemen and Jager (1997). However, it is impossible to review all approaches and works related to mixed performance problems. Hence only some most related work will be described here. Bernstein and Haddad (1989) proposed for the first time the mixed $H_2/H_\infty$ as a way to formulate a meaningful optimization problem to the standard $H_\infty$ control problem by using the Lagrange multiplier method. Later on, Khargonekar and Rotea (1991), and Halder et al (1997) addressed the mixed $H_2/H_\infty$ problem and converted the state feedback mixed design into a convex optimization problem. A time domain signal formulation for a dual mixed $H_2/H_\infty$ problem was presented and characterized by Doyle et al (1994) and Zhou et al (1990,1994). In a different direction, Limebeer et al (1994) and Chen and Zhou (2001) considered Nash game approaches to these mixed problems. Solutions based on linear matrix inequalities (LMI) to multi-objective problems have also been proposed for output feedback control by Oliveira et al (1999), and Scherer et al (1997). In this design, the objectives
are formulated in terms of a common Lyapunov function. However, this formulation tends to be very conservative. Clement and Duc (2000) presented an extension to this method in order to use different Lyapunov functions for each design objective. Hindi et al (1998), and Scherer (1995, 2000) proposed to combine LMI’s with the Youla parameterization in order to search for the optimal Youla parameter in a finite dimensional space. Similarly, Qi et al (2001) proposed finite dimensional approximations for the Youla parameter in order to solve mixed problems with time-domain contraints. Thus, an optimal solution is reached by approximating it from below and above. Due to its physical interpretation, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design has also been applied to the optimal filtering problem, e.g. (Khargonekar et al 1996, Rotstein et al 1996). Multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ problems have also been characterized in terms of their duality description (Djouadi et al 2001). Furthermore, Chen et al (2000) extended the $\mathcal{H}_2/\mathcal{H}_\infty$ design to nonlinear systems by using a fuzzy output feedback controller.

Another mixed problem, the $\mathcal{L}_1/\mathcal{H}_\infty$ control was synthesized via convex optimization of finite dimensional approximations by Sznaier and Bu (1998). Haddad et al (1998) revisited the $\mathcal{L}_1/\mathcal{H}_\infty$ problem, but now a fixed structure controller is suggested and optimality conditions are derived for its solution. Sznaier and Blanchini (1994) proposed the design of rational mixed $\mathcal{L}_1/\mathcal{H}_\infty$ suboptimal controllers by solving a sequence of finite dimensional auxiliary problems. On the other hand, solutions to the $\mathcal{H}_2/\mathcal{L}_1$ problems were presented by Amishima et al (1988), Voulgaris (1994), and Wu and Chu (1999) where quadratic programming problems were introduced to reach a solution. In addition, Salapaka and Khammash (1998) and Salapaka et al (1999) approached the $\mathcal{H}_2/\mathcal{L}_1$ problem by obtaining upper and lower bound convergence methods that involve the $Q$ parameter in the Youla parameterization. Sznaier and Amishima (1998) studied the $\mathcal{H}_2$ problem with time constraints and suggested approximations to the optimal solutions by solving some quadratic programming problems. In general, most of approaches presented in the literature do not solve directly the true mixed performance problem by either optimizing an upper bound of the true performance or using a related performance criterion since the original mixed performance problem is a highly complex and nonlinear constrained optimization problem.

In recent years, evolutionary schemes have been extensively used to solve nonlinear constrained optimization problems where multi-local minima can restrict global convergence. Evolutionary schemes are inspired by the natural selection criteria where the stronger organisms are likely to survive after generations. Thus, a parallelism can be drawn with an optimization problem where the evolution period is considered as the optimization time and the most fitted organism in the population will represent the optimal solution. Two evolutionary schemes, evolution algorithms and genetic algorithms (Pham and Karaboga 2000), are most commonly used. These algorithms
present two main characteristics: a multi-directional (random) search and an information exchange among best solutions. These properties can generate new search directions in order to avoid local minima. Applications of genetic algorithms to control and signal processing have been reported in literature: digital IIR filter design (Man et al 1999), adaptive recursive filtering (White and Flockton 1997), active noise control (Tang et al 1995), systems model reduction (Li et al 1997), weighting function design for $H_\infty$ loop-shaping (Whidborne et al 1995), etc.

Motivated from those successful applications of the evolutionary algorithms, it seems nature to apply these algorithms such as genetic algorithms to the above mentioned multiobjective optimization problems. Nevertheless, it is unlikely to produce any reasonable results if these algorithms are applied blindly since the optimization parameter spaces are too large. In this paper, we shall propose design techniques that explicitly parameterize the free transfer matrix $Q$ in the $H_\infty$ and $H_2$ suboptimal controller parameterizations such that the optimization parameter spaces are highly restricted and then evolutionary algorithms such as genetic/evolution algorithms can be applied effectively to produce the desired results. In addition to the computational advantage, the proposed technique may produce much lower order controllers than those using Youla parameterization and convex optimization.

The rest of the paper is organized as follows. First, notations and some definitions used in the paper are presented in Section 2. Next, the suboptimal $H_\infty$ and $H_2$ controller parameterizations are introduced in Section 3. Section 4 gives some explicit parameterizations of norm bounded $H_\infty$ and $H_2$ functions. In Section 5, various multi-objective design problems are presented and optimization schemes are proposed. Finally, some numerical examples are shown in Section 6 and the paper is concluded with some remarks in Section 7.

## 2 Notations and Definitions

Let $G(s)$ be an MIMO transfer matrix with the following state-space realization

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} G_{11}(s) & \ldots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{p1}(s) & \ldots & G_{pm}(s) \end{bmatrix}$$

(1)

Let $H_2$ denote the space of all strictly proper and stable transfer matrices. The $H_2$ norm is defined as

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}[G^*(j\omega)G(j\omega)]d\omega$$

(2)
It can be computed by using the state space representation of $G$. For a stable and strictly proper $G$ (i.e. $A$ is stable and $D = 0$), we have

$$\|G\|_2^2 = \text{Trace}(B^* L_o B) = \text{Trace}(CL_c C^*)$$

(3)

where $L_c$ and $L_o$ are the controllability and observability Gramians which can be obtained by solving the following Lyapunov equations

$$AL_c + L_c A^* + BB^* = 0 \quad A^* L_o + L_o A + C^* C = 0$$

(4)

Let $\mathcal{RH}_\infty$ denote the space of all proper and real rational stable transfer functions. The $\mathcal{H}_\infty$ norm is defined as:

$$\|G\|_\infty = \sup_{\text{Re}(s) > 0} \bar{\sigma}[G(s)] = \sup_{\omega \in \mathcal{R}} \bar{\sigma}[G(j\omega)]$$

(5)

The $L_1$ norm of a stable transfer matrix is defined as

$$\|G\|_1 = \max_{1 \leq i \leq p} \sum_{j=1}^{m} \|g_{ij}\|_1$$

(6)

where $g_{ij}(t) = \mathcal{L}^{-1}\{G_{ij}(s)\}$ and

$$\|g_{ij}\|_1 = \int_{0}^{\infty} |g_{ij}(t)| dt$$

(7)

Consider a feedback system described by the block diagram in Figure 1 where the generalized plant $G$ and the controller $K$ are assumed to be real-rational and proper with $y(t) \in \mathcal{R}^{p_2}$ and $u(t) \in \mathcal{R}^{m_2}$. Let $G$ be partitioned accordingly as

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

(8)

and

$$K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

Then the transfer function from $w$ to $z$ is given by

$$T_{2w} = \mathcal{F}_l (G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$$

(9)

where

$$A_{cl} = \begin{bmatrix} A + B_2 \bar{R}^{-1}D_k C_2 & B_2 \bar{R}^{-1}C_k \\ B_k R^{-1}D_{21} & A_k + B_k R^{-1}D_{22}C_k \end{bmatrix}$$
\[
B_{cl} = \begin{bmatrix}
B_1 + B_2 \tilde{R}^{-1} D_k D_{21} \\
B_k R^{-1} D_{21}
\end{bmatrix}
\]
\[
C_{cl} = \begin{bmatrix}
C_1 + D_{12} D_k R^{-1} C_2 \\
D_{12} \tilde{R}^{-1} C_k
\end{bmatrix}
\]
\[
D_{cl} = D_{11} + D_{12} D_k R^{-1} D_{21}
\]
\[
R = I - D_{22} D_k, \quad \tilde{R} = I - D_k D_{22}
\]

and \( \mathcal{F}_I(\cdot, \cdot) \) is called a lower linear fractional transformation.

### 3 \( \mathcal{H}_\infty \) and \( \mathcal{H}_2 \) Suboptimal Controller Parameterizations

Consider the generalized feedback system described in Figure 1 with the generalized plant \( G \) given by (8) with some suitable assumptions. Then it is well known that all stabilizing controllers \( K(s) \) satisfying the \( \mathcal{H}_\infty \) condition, \( ||T_{zw}||_\infty < \gamma \) for a given \( \gamma > 0 \), can be parameterized as \( K = \mathcal{F}_I(M_\infty, Q) \) with \( Q \in \mathcal{RH}_\infty, ||Q||_\infty < \gamma \) where

\[
M_\infty = \begin{bmatrix}
\hat{A} & \hat{B}_1 & \hat{B}_2 \\
\hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\
\hat{C}_2 & \hat{D}_{21} & \hat{D}_{22}
\end{bmatrix} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

and \( M_\infty \) is constructed from the solutions of two Riccati equations (Doyle et al 1999, Zhou et al 1996, Zhou and Doyle 1997). In the case of \( Q = 0 \), the solution, \( K = M_{11} \), is called the central controller. Note that there is no guarantee that \( M_\infty \) is itself stable even though the closed-loop system is stable. It is noted that \( M_\infty \) and \( Q \) are \( (p_2 + m_2) \times (p_2 + m_2) \) and \( m_2 \times p_2 \) transfer matrices respectively.

To consider the \( \mathcal{H}_2 \) optimal control problem, we shall assume for simplicity that \( D_{11} = 0 \) and \( D_{22} = 0 \). The case where \( D_{22} \neq 0 \) can be dealt with easily. It is also well known that all stabilizing controllers for the generalized plant \( G \) can be written as \( K = \mathcal{F}_I(M_2, Q) \) with

\[
M_2 = \begin{bmatrix}
\hat{A}_2 & -L_2 & B_2 \\
F_2 & 0 & I \\
-C_2 & I & 0
\end{bmatrix}
\]

where \( \hat{A}_2 = A + B_2 F_2 + L_2 C_2 \) and \( F_2 \) and \( L_2 \) can be constructed from the solutions of two related Riccati equations, see (Zhou et al 1996, Zhou and Doyle 1997). It is then clear that the closed-loop \( \mathcal{H}_2 \) norm is given by

\[
||T_{zw}||_2^2 = ||G_c B_1||_2^2 + ||F_2 G_f||_2^2 + ||Q||_2^2
\]
where the definition of the transfer matrices $G_c$ and $G_f$ can be found in (Zhou et al 1996, Zhou and Doyle 1997). Consequently, $Q = 0$ represents the optimal solution to the $\mathcal{H}_2$ problem. As in the $\mathcal{H}_\infty$ parameterization, $M_2$ and $Q$ are $(p_2 + m_2) \times (p_2 + m_2)$ and $m_2 \times p_2$ transfer matrices respectively. Let $\gamma_{opt}^2 = \|G_c B_1\|^2 + \|F_2 G_f\|^2$. Then for any $\gamma > \gamma_{opt}$, all suboptimal $\mathcal{H}_2$ controllers satisfying $\|T_{zw}\|_2 < \gamma$ can be parameterized as $K = F_1(M_2, Q)$ with $Q \in \mathcal{R} \mathcal{H}_2$ and $\|Q\|^2 < \gamma^2 - \gamma_{opt}^2$.

4 Parameterizations of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Norm Bounded Functions

It is now clear that if a controller is required to satisfy both the $\mathcal{H}_\infty$ norm constraint, $\|T_{zw}\|_\infty < \gamma$, and some additional performance objectives, it has to come from the family of $\mathcal{H}_\infty$ controllers parameterized in the last section. In other words, a stable $Q$ with $\|Q\|_\infty < \gamma$ must be found to satisfy the additional performance objectives. Similar observations can be made for problems involving $\mathcal{H}_2$ performance objectives. To find a suitable $Q \in \mathcal{R} \mathcal{H}_\infty$ with $\|Q\|_\infty < \gamma$ or a $Q \in \mathcal{R} \mathcal{H}_2$ with $\|Q\|_2 < \sqrt{\gamma^2 - \gamma_{opt}^2}$, it is desirable to have more explicit characterizations of these norm bounded analytic functions that are appropriate for numerical optimization.

Stein and Bosgra (1991) presented a parameterization for an $\mathcal{H}_\infty$ norm bounded strictly proper and stable transfer matrix. That result was extended to the proper case (Campos and Zhou 2002) by the following lemma.

Lemma 1 Let $\gamma > 0$ and let $Q$ be a stable transfer matrix of degree $n_q$ and $\|Q\|_\infty < \gamma$. Then $Q$ can be represented as $Q = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix}$ with $A_q = A_{q_k} + A_{q_\ast}$ for some $A_{q_k} = -A_{q_k}^\ast \in \mathcal{R}^{n_q \times n_q}$, $B_q \in \mathcal{R}^{n_q \times p_2}$, $C_q \in \mathcal{R}^{m_2 \times n_q}$, $D_q \in \mathcal{R}^{m_2 \times p_2}$, and

$$A_{q_k} = \frac{1}{2} \left( -B_q R^{-1} D_q^* C_q - C_q^* D_q R^{-1} B_q^* - B_q R^{-1} B_q^* - C_q^* (I + D_q R^{-1} D_q^*) C_q \right)$$

$$\bar{\sigma}(D_q) < \gamma$$

where $R = \gamma^2 I - D_q^* D_q$.

Proof: Assume that $Q = \begin{bmatrix} \hat{A}_q & \hat{B}_q \\ \hat{C}_q & \hat{D}_q \end{bmatrix} \in \mathcal{R} \mathcal{H}_\infty$ is a $n_q^{th}$ order observable realization and $\|Q\|_\infty < \gamma$, then according with the Bounded Real Lemma (Zhou et al 1996) $\bar{\sigma}(D_q) < \gamma$ and $\exists Y > 0$ such that

$$Y(\hat{A}_q + \hat{B}_q R^{-1} D_q^* \hat{C}_q) + (\hat{A}_q + \hat{B}_q R^{-1} D_q^* \hat{C}_q)^* Y + Y \hat{B}_q R^{-1} \hat{B}_q^* Y + \hat{C}_q^* (I + D_q R^{-1} D_q^*) \hat{C}_q = 0$$

where $R = \gamma^2 I - D_q^* D_q$. Since $Y > 0$, there exists a Cholesky factorization of $Y = T^* T$. Now $T$ is
invertible and can be used as a similarity transformation on $Q$

$$Q = \begin{bmatrix} T\hat{A}_q T^{-1} & T\hat{B}_q \\ \hat{C}_q T^{-1} & D_q \end{bmatrix} = \begin{bmatrix} A_q & B_q \\ C_q & D_q \end{bmatrix}$$

(16)

Thus, the Riccati equation in (15) becomes

$$A_q + A_q^* + B_q R^{-1} D_q^* C_q + C_q^* D_q R^{-1} B_q^* + B_q R^{-1} B_q^* + C_q^* (I + D_q R^{-1} D_q^*) C_q = 0$$

(17)

Furthermore, $A_q$ can be decomposed into a symmetric part $A_{qs}$ and a skew symmetric part $A_{qk}$ where

$$A_{qs} = (A_q + A_q^*)/2 \quad A_{qk} = (A_q - A_q^*)/2$$

(18)

Consequently, the skew symmetric part $A_{qk}$ disappears from (17) and the result in (13) is obtained.

\[ \square \]

Note that when $D_q = 0$, we have (Stein and Bosgra, 1991)

$$A_{qs} = -\frac{1}{2} (B_q B_q^*/\gamma^2 + C_q^* C_q)$$

(19)

On the other hand, using the definition of the $\mathcal{H}_2$ norm given by (3) and (4), a parameterization for all $Q \in \mathcal{RH}_2$ follows (Campos and Zhou 2002).

**Lemma 2** Assume that $Q \in \mathcal{RH}_2$ has degree $n_q$, then $Q$ can be represented in the following form

$$\|Q\|_2^2 = \text{Trace}(B_q^* B_q)$$

with $Q = \begin{bmatrix} A_q & B_q \\ C_q & 0 \end{bmatrix}$ and $A_q = A_{qs} + A_{qk}$ where

$$A_{qs} = -\frac{1}{2} C_q^* C_q, \quad A_{qk} = -A_{qk}^* \in \mathbb{R}^{n_q \times n_q}, \quad B_q \in \mathbb{R}^{n_q \times p_2}, \quad C_q \in \mathbb{R}^{m_2 \times n_q}.$$ 

(20)

**Proof:** Assume that $Q = \begin{bmatrix} \hat{A}_q & \hat{B}_q \\ \hat{C}_q & 0 \end{bmatrix} \in \mathcal{RH}_2$ is a $n_q^{th}$ order observable realization, then according to (3) and (4), $\exists Y > 0$ such that

$$\hat{A}_q^* Y + Y \hat{A}_q + \hat{C}_q^* \hat{C}_q = 0$$

(21)

Since $Y > 0$, there exists a Cholesky factorization of $Y = T^* T$. Now $T$ is invertible and can be used as a similarity transformation on $Q$

$$Q = \begin{bmatrix} T\hat{A}_q T^{-1} & T\hat{B}_q \\ \hat{C}_q T^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A_q & B_q \\ C_q & 0 \end{bmatrix}$$

(22)
Thus, the Lyapunov equation in (21) becomes
\[ A_q + A_q^* + C_q^* C_q = 0 \] (23)
Moreover, \( A_q \) can be decomposed into a symmetric part \( A_q^s \) and a skew symmetric part \( A_q^k \). Consequently, the skew symmetric part \( A_q^k \) disappears from (23) and it is obtained
\[ \|Q\|^2_2 = \text{Trace}(B_q^* B_q) \]
\[ A_{qs} = -\frac{1}{2} C_q^* C_q \] (24)
Moreover, following the result of the previous lemma, \( \|Q\|_2 < \beta \) is equivalent to \( \text{Trace}(B_q^* B_q) < \beta^2 \). On the other hand, note that if some \( Q \) is constructed according with (20), it is possible that the resulting transfer function \( Q \) had eigenvalues on the imaginary axis. However, only those eigenvalues will be unobservable and the remaining observable ones will be located in the LHP. Consequently, the resulting \( Q \) still belongs to \( \mathcal{RH}_2 \). This will be proved in the next lemma.

**Lemma 3** If \( Q \) has degree \( n_q \) and is constructed following (20), then \( Q \in \mathcal{RH}_2 \).

**Proof:** \( Q = \begin{bmatrix} A_q & B_q \\ C_q & 0 \end{bmatrix} \) and by the construction of (20), the lyapunov equation
\[ A_q^* L_o + L_o A_q + C_q^* C_q = 0 \] (25)
has a solution \( L_o = I \). Now, let \( \lambda \) be an eigenvalue of \( A_q \) and \( v \neq 0 \) be a corresponding eigenvector \( (A_q v = \lambda v) \). Pre-multiply (25) by \( v^* \) and post-multiply (25) by \( v \) to obtain
\[ 2\Re\{\lambda v^* v\} + v^* C_q^* C_q v = 0 \] (26)
Since \( C_q^* C_q \geq 0 \), then it is clear that \( \Re\{\lambda\} \leq 0 \). Moreover, if \( \Re\{\lambda\} = 0 \) (imaginary axis mode), then \( C_q v = 0 \). Consequently, the imaginary axis modes are unobservable. However, the remaining LHP eigenvalues \( (\Re\{\lambda\} < 0) \) are observable \( (C_q v \neq 0) \). As a result, \( Q \in \mathcal{RH}_2 \).

\[ \square \]

## 5 Mixed \( \mathcal{L}_1/\mathcal{H}_2/\mathcal{H}_\infty \) Performance Problems

Several mixed objective control problems will be presented in this section. Their solutions are critically dependent upon the \( \mathcal{H}_\infty \) and \( \mathcal{H}_2 \) controller state-space parameterizations presented in the previous sections. That is, the parameters of the transfer matrix \( Q \) will be used to optimize the performance indices and satisfy constraints. Note that since the controllers are based on the \( \mathcal{H}_2 \)
and $\mathcal{H}_\infty$ parameterizations the internal stability of the closed-loop is always guaranteed. Assume that the degree of $Q$ is predefined to $n_q$, then according with the dimensions of the generalized plant (8), the number of variables of each component of $Q$ is given by

$$A_{q_k} = \begin{bmatrix}
0 & a_{12} & a_{13} & \cdots & a_{1n_q} \\
-a_{12} & 0 & a_{23} & \cdots & a_{2n_q} \\
-a_{13} & -a_{23} & 0 & \cdots & a_{3n_q} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{1n_q} & -a_{2n_q} & -a_{3n_q} & \cdots & 0
\end{bmatrix} \Rightarrow \frac{(n_q - 1)n_q}{2} \quad (27)$$

$$B_q = \begin{bmatrix}
b_{11} & \cdots & b_{1p_2} \\
\vdots & \ddots & \vdots \\
b_{n_q1} & \cdots & b_{n_qp_2}
\end{bmatrix} \Rightarrow n_q \times p_2 \quad (28)$$

$$C_q = \begin{bmatrix}
c_{11} & \cdots & c_{1n_q} \\
\vdots & \ddots & \vdots \\
c_{m_{21}} & \cdots & c_{m_{2n_q}}
\end{bmatrix} \Rightarrow m_2 \times n_q \quad (29)$$

$$D_q = \begin{bmatrix}
d_{11} & \cdots & d_{1p_2} \\
\vdots & \ddots & \vdots \\
d_{m_{21}} & \cdots & d_{m_{2p_2}}
\end{bmatrix} \Rightarrow m_2 \times p_2 \quad (30)$$

Since $Q \in \mathcal{RH}_\infty$ is an $m_2 \times p_2$ transfer matrix, the total number of variables in the optimization scheme will be $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2 + m_2 p_2$. In order to solve the multi-objective problems, the elements of the state-space description of $Q$ are aligned into a vector form

$$X = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{(n_q-1)n_q} & b_{11} & \cdots & b_{n_qp_2} & c_{11} & \cdots & c_{m_{2n_q}} & d_{11} & \cdots & d_{m_{2p_2}}
\end{bmatrix}^T \quad (31)$$

Therefore, the proposed optimization problems will be solved with respect to the variable vector $X$. Note that if $Q \in \mathcal{RH}_2$, i.e. $D_q = 0$, the number of free variables reduces to $(n_q - 1)n_q/2 + n_q m_2 + n_q p_2$. During the optimization, an interval of variation will be set for all the parameters in $X$, i.e.

$$a_{min} \leq a_{ij} \leq a_{max}$$

$$b_{min} \leq b_{ij} \leq b_{max}$$

$$c_{min} \leq c_{ij} \leq c_{max}$$

$$d_{min} \leq d_{ij} \leq d_{max}$$

(32)

However, not all the requirements in the elements of $Q$ can be reflected into a range of variation for each parameter. Therefore, the following penalty functions are used in the optimization schemes.
to restrict the variations of some of the parameters and to enforce requirements on the transfer matrix \( Q \)

\[
P(D, \gamma) = \begin{cases} 
M & \bar{\sigma}(D) \geq \gamma \\
1 & \text{otherwise} \end{cases} 
\] (33)

\[
J(B, \gamma, \gamma_{opt}) = \begin{cases} 
N & \text{Trace}(B^*B) \geq \gamma^2 - \gamma_{opt}^2 \\
1 & \text{otherwise} \end{cases} 
\] (34)

where the matrices \( A, B \) and \( D \) are linked to the state-space realization of a system, \( \gamma > \gamma_{opt} > 0 \), \( M \) and \( N \) are constants \( \gg 1 \).

Consider a generalized feedback system described in figure 2 with

\[
G_m = \begin{bmatrix}
A & B_0 & B_1 & B_2 \\
C_0 & D_{00} & D_{01} & D_{02} \\
C_1 & D_{10} & D_{11} & D_{12} \\
C_2 & D_{20} & D_{21} & D_{22}
\end{bmatrix}, \quad \begin{bmatrix}
z_0 \\
z \\
y
\end{bmatrix} = G_m \begin{bmatrix}
w_0 \\
w \\
u
\end{bmatrix} 
\] (35)

Let \( G_0 \) and \( G \) be defined as

\[
G_0 = \begin{bmatrix}
A & B_0 & B_2 \\
C_0 & D_{00} & D_{02} \\
C_2 & D_{20} & D_{22}
\end{bmatrix}, \quad G = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} 
\] (36)

Then, the next closed-loop transfer matrices can be determined

\[
T_{z_0 w_0} = \mathcal{F}_l (G_0, K), \quad T_{zw} = \mathcal{F}_l (G, K) 
\] (37)

## 5.1 Mixed \( \mathcal{H}_\infty/\mathcal{H}_\infty \) Design (Robust \( \mathcal{H}_\infty \) Performance)

The robust performance problem can be formulated as

\[
\min_{\text{stabilizing } K} \| T_{z_0 w_0} \|_\infty \quad \text{s.t.} \quad \| T_{zw} \|_\infty < \gamma 
\] (38)

for some \( \gamma > \gamma_\infty \) where

\[
\gamma_\infty = \min_{\text{stabilizing } K} \| T_{zw} \|_\infty 
\] (39)

In order to solve (38), the following numerical optimization is proposed

\[
\min_{A_{q_k}, B_q, C_q, D_q} P(D_q, \gamma) : \| \mathcal{F}_l (G_0, \mathcal{F}_l (M_\infty, Q)) \|_\infty 
\] (40)

where \( Q = \begin{bmatrix} A_{q_k} + A_{q_s} & B_q \\ C_q & D_q \end{bmatrix} \), \( M_\infty \) is given by (10) and \( A_{q_s} \) by (13). Therefore, \( (A_{q_k}, B_q, C_q, D_q) \) are the free variables in the optimization scheme. The penalty function \( P(\cdot, \cdot) \) is included to restrict
the maximum singular value of $D_q$ to be $< \gamma$, which is a necessary condition in the Bounded Real Lemma to have $\|Q\|_\infty < \gamma$. This condition could not be incorporated in the optimization directly since it is difficult to give an interval of variation for the elements of a matrix to have $\bar{\sigma} < \gamma$. So it was decided to introduce the penalty function in the cost function to detect violations of the condition and penalize these solutions. The result obtained from the optimization in (40) is finally used to construct the multi-objective controller as $K = F_l(M_\infty, Q)$.

5.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$

This mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design problem can be formulated as

$$\min_{\text{stabilizing } K} \|T_{z_0w_0}\|_2 \quad \text{s.t.} \quad \|T_{zw}\|_\infty < \gamma$$

(41)

for some $\gamma > \gamma_\infty$. In order to solve (41), the following numerical optimization is proposed

$$\min_{A_{q_k},B_q,C_q} \|{\mathcal{F}_l}(G_0, {\mathcal{F}_l}(M_\infty, Q))\|_2$$

(42)

where $Q = \begin{bmatrix} A_{q_k} + A_{q_s} & B_q \\ C_q & 0 \end{bmatrix}$ since now $Q \in \mathcal{RH}_2$, $M_\infty$ is given by (10) and $A_{q_s}$ by (13). Note that a penalty function is not needed since, $Q$ has to be strictly proper, i.e. $D_q = 0$. Consequently, in this mixed problem, the free variables in the optimization are now $(A_{q_k}, B_q, C_q)$.

5.3 Mixed $\mathcal{L}_1/\mathcal{H}_\infty$

The mixed $\mathcal{L}_1$ and $\mathcal{H}_\infty$ design problem can be stated as

$$\min_{\text{stabilizing } K} \|T_{z_0w_0}\|_1 \quad \text{s.t.} \quad \|T_{zw}\|_\infty < \gamma$$

(43)

for some $\gamma > \gamma_\infty$. In order to solve (43), the following numerical optimization is proposed

$$\min_{A_{q_k},B_q,C_q,D_q} P(D_q, \gamma) \cdot \|{\mathcal{F}_l}(G_0, {\mathcal{F}_l}(M_\infty, Q))\|_1$$

(44)

where $M_\infty$ is given by (10), $A_{q_s}$ by (13) and $Q = \begin{bmatrix} A_{q_k} + A_{q_s} & B_q \\ C_q & D_q \end{bmatrix}$. Note that the penalty function $P(\cdot, \cdot)$ is needed again to restrict $\bar{\sigma}(D_q) < \gamma$ since $Q \in \mathcal{RH}_\infty$. Therefore, $(A_{q_k}, B_q, C_q, D_q)$ are the free variables in the optimization scheme.

5.4 $\mathcal{L}_1/\mathcal{H}_2$ Design

The mixed $\mathcal{L}_1/\mathcal{H}_2$ problem is defined as:

$$\min_{\text{stabilizing } K} \|T_{z_0w_0}\|_1 \quad \text{s.t.} \quad \|T_{zw}\|_2 < \gamma$$

(45)
for some $\gamma > \gamma_{\text{opt}}$ where
\begin{equation}
\gamma_{\text{opt}} = \min_{\text{stabilizing } K} \| T_{zw} \|_2
\end{equation}

In order to solve (45), the following numerical optimization is proposed
\begin{equation}
\min_{A_{q_k}, B_q, C_q, D_q} \; J(B_q, \gamma, \gamma_{\text{opt}}) \cdot \| F_l(G_0, F_l(M_2, Q)) \|_1
\end{equation}

where $A_{q_s} = -C_q^*C_q/2$ and $Q = \begin{bmatrix} A_{q_k} + A_{q_s} & B_q \\ C_q & 0 \end{bmatrix}$. Therefore, $(A_{q_k}, B_q, C_q)$ are the free variables in the optimization scheme with $\| Q \|_2^2 < \gamma^2 - \gamma_{\text{opt}}^2$. The penalty function $J(\cdot, \cdot, \cdot)$ is introduced here to enforce the restriction on the norm of $Q$, i.e. limit the value of the elements of $B_q$ to satisfy $\text{Trace}(B_q^*B_q) < \gamma^2 - \gamma_{\text{opt}}^2$. Note that this condition cannot be translated into any pattern selection for the elements of $B_q$. Therefore, it was chosen to include the penalty function in the cost function to penalize any combination of parameters that violate it.

The optimization problems just presented in (40), (42), (44) and (47) are generally nonconvex and present intrinsically multi-modal characteristics. Therefore, it is natural to think that these problems are not practical to solve. Nevertheless, evolution optimization has turned out to be quite useful to solve these complicated optimization problems. Moreover, if it is used in conjunction with a well-known gradient-based optimization technique, a powerful optimization scheme is obtained that can effectively solve the proposed problems.

6 Numerical Examples

In this section, the proposed mixed performance problems will be illustrated through some numerical examples. Due to space limit, we shall only include two examples here. In all these examples, the first stage of the optimization scheme was carried out by a genetic/evolution algorithm (Pham and Karaboga 2000). The second stage was carried out by a quasi-newton plus linear search scheme (Grace 1992, Nocedal and Wright 1999). In this way, the genetic/evolution algorithm was applied first to perform a global search in the parameters space and find a minimum solution. Next, a local search was conducted to obtain the optimal solution. This two-stage optimization outperformed the use of each one of the algorithms alone. In order to implement the first stage, two codes were generated in MATLAB as m-files: EVAOCP (Evolution Algorithm for Optimization of Continuous Parameters), and GAOCP (Genetic Algorithm for Optimization of Continuous Parameters). The gradient-based optimization was carried out by using the Optimization Toolbox (Grace 1992) of MATLAB.
So far, there is no direct computation of the $L_1$ norm except its mathematical definition (6). Consequently, in order to approach the $L_1/H_\infty$ (44) or $L_1/H_2$ (47) designs, the norm computation was approximated. Note that the impulse response of each element of (1) is given by

$$g_{ij}(t) = C_i e^{A_t} B_j + D_{ij} \delta(t)$$

(48)

where $B = [B_1 \ldots B_m]$, $C = [C_1 \ldots C_p]^*$ and $D = [D_{ij}]$. Hence, the infinite integration of (7) can be approximated according with the poles of $G_{ij}(s)$, estimating the time such that $g_{ij}(t)$ is below certain percentage of its peak value and performing the integration in that interval of time.

The following numerical examples were computed on a PC Pentium III at 933 MHz. The constants $M$ in (33) was given the value $1 \times 10^6$.

6.1 Example 1

This example is taken from Baeyens and Khargonekar (1994). The multi-objective problem is a mixed $H_2/H_\infty$ design with the generalized plant $G_m$ given by

$$G_m = \begin{bmatrix} A & B_1 & B_2 \\ C_0 & 0 & D_0 \\ C_1 & 0 & D_1 \\ C_2 & D_2 & 0 \end{bmatrix}$$

(49)

where the description of the matrices $(A, B_1, B_2, C_0, C_1, C_2)$ is given in (Baeyens and Khargonekar 1994). This is a special case of the mixed $H_2/H_\infty$ problems formulated in the previous sections with $w_0 = w$.

Thus the mixed problem is formulated as

$$\min_{Kstabilizing} \|T_{z*w}\|_2 \quad s.t. \quad \|T_{z*w}\|_\infty < \gamma$$

(50)

The optimization scheme in (42) was used to obtain the mixed controller. In order to judge the performance of the proposed scheme, the mixed $H_2/H_\infty$ optimization was also carried out using the LMI Toolbox (Gahinet et al 1995) of MATLAB©. In this example, the results reported for the proposed method (42) are obtained by using an evolution algorithm (EVAOCP) in the first stage of the optimization. The results were almost identical if a genetic algorithm was used instead. The example was run for two values of $\gamma$: 1.6 and 2.0. Note that the closed-loop performance $\|T_{z*w}\|_\infty < \gamma$ is always guaranteed by the proposed optimization algorithm (42).

The computational effort and performance for different order $Q$s was investigated for $\gamma = 1.6$. The algorithm was evaluated 5 consecutive times for the same order of $Q$. Table 1 presents these
results. The mean value and standard deviation (sdt) for each respective order of the floating point operations (flops), computation time, and resulting performance $\|T_{zw}\|_2$ are presented. Therefore, it is seen that there is no significant performance improvement by choosing a $Q$ of order higher than 2\textsuperscript{nd}. Also, it is clear that the algorithm reached almost constantly the same performance level ($\|\cdot\|_2$) for each order, since the standard deviation is close to zero for all cases. It is also evident that the computational cost is increased by raising the $Q$ order. Figure 3 shows the flops required during the optimization as a function of the $Q$ order.

Table 2 summarize the results obtained for $\gamma = 1.6$ and 2.0. A 2\textsuperscript{nd} order $Q$ is needed to provide these results. Thus, the optimization was carried out for 5 parameters ($Q$ is strictly proper). In the first stage, the $\mathcal{H}_2$ cost was 23.04 and 4.72 for $\gamma = 1.6$ and 2.0 respectively. In the second stage, these values were reduced further to 22.875 and 4.53. Figure 4 presents the evolution of the cost function for $\gamma = 2.0$ during the first stage.

### 6.2 Example 2

This example is taken from Sznaier and Blanchini (1994). The mixed $\mathcal{L}_1/\mathcal{H}_\infty$ design is to solve the following minimization

$$\min_{K_\text{stabilizing}} \|T_{zw}\|_1 \quad \text{s.t.} \quad \|T_{zw}\|_\infty < \gamma$$  \hfill (51)

The generalized plant $G$ is given by

$$G = \begin{bmatrix} 0 & \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \\ W_2 & \begin{bmatrix} W_2P \\ W_2P \end{bmatrix} \\ -1 & -P \end{bmatrix}$$  \hfill (52)

where $W_1$, $W_2$ and $P$ are defined in (Sznaier and Blanchini 1994). The value of $\gamma$ was set to 2.6 as in the original paper. Consequently, the optimization proposed in (44) was applied in order to compute the mixed controller. For this example, a genetic algorithm (GAOCP) was used in the first stage of the optimization. However, no clear advantages or disadvantages are shown using either algorithm.

Similarly to the first example, the computational effort and performance for different order $Q$s was first investigated ($\gamma = 2.6$). Table 3 presents these results varying the $Q$ order among 1\textsuperscript{st} and 3\textsuperscript{rd}. No significant performance improvements were observed by using a $Q$ of higher order. The algorithm was run again 5 consecutive times for the same order of $Q$. Analyzing table 3, it is observed that the algorithm reached almost constantly the same performance level ($\|\cdot\|_1$) for any order, since the standard deviation is always close to zero. However, the computational cost varied
drastically for all cases. In addition, the computational time and flops are constantly increased by raising the $Q$ order. Figure 5 shows the flops required during the optimization as a function of the $Q$ order.

Table 4 summarize these result with a 2\textsuperscript{nd} order $Q$. Hence, a 6 parameter optimization was carried out ($Q$ is now a proper transfer matrix). After the first stage optimization, the optimal cost was 4.1216. The second stage of the optimization reduced this value to 4.04 (i.e. $\|T_{zu}\|_1 = 4.04$). Furthermore, the value of the $H_\infty$ norm is $\|T_{zu}\|_\infty = 2.5476 < 2.6$. On average, 307.77 sec. and $6.66 \times 10^9$ flops were needed to reach a solution. In this case, the computation of the $L_1$ norm slowed down the optimization scheme. However, an improvement of $L_1$ performance is seen compared with the result by Sznaier and Blanchini (1994). In figure 6, the evolution of the cost function is presented for the first stage of the optimization scheme.

7 Conclusions

Optimization schemes are presented to solve the multi-objective design problems. Parameterizations of norm bounded $H_\infty$ and $H_2$ functions were used to limit the number of variables and restrict the optimizations. This step reduces the complexity of the mixed synthesis problem and guarantees the closed-loop $H_\infty$ or $H_2$ performance according with the selected parameterization. Thus, the search for the appropriate $Q$ parameter looks to minimize another performance measure. The resulting optimizations with this method are highly nonlinear and present multi-modal characteristics. For this reason, a two-stage algorithm was used in the optimization process. Numerical examples show the success of the optimization schemes to design mixed controllers. However, it is not possible to establish the best achievable performance with these techniques and this issue has to be explored iteratively. It is interesting to note that only low order $Q$s were needed in the numerical examples. Thus, the orders of resulting controllers are comparable to those of the generalized plants. In all the benchmark examples, the performance was always improved compared to previous results published in the literature. It should be pointed out that more complex mixed problems can also be treated in the same framework without any additional difficulty.
References


Campos-Delgado, D.U., and Zhou, K., 2002, $H_\infty$ and $H_2$ strong stabilization by numerical optimization. 15$^{th}$ IFAC World Congress, Barcelona, Spain, 21-26 June.


Clement, B., and Duc, G., 2000, Flexible arm multiobjective control via youla parameterization and LMI optimization. 3$^{rd}$ IFAC Symposium on Robust Control Design, Prague, Czech Republic, 21-23 June.


Salapaka, M.V., and Khammash, M., 1998, Multi-objective MIMO optimal control


Zhou, K., Doyle, J.C., Glover, K., and Bodenheimer, B., 1990, Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theory. *American Control Conference*, San Diego, USA, pp. 2502-2507.


20
<table>
<thead>
<tr>
<th>$Q$ order</th>
<th>Flops</th>
<th>Computation Time (sec.)</th>
<th>$|T_{z_\theta w}|_2$</th>
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<tbody>
<tr>
<td></td>
<td>mean</td>
<td>std</td>
<td>mean</td>
</tr>
<tr>
<td>1st</td>
<td>$1.38 \times 10^8$</td>
<td>$3.33 \times 10^5$</td>
<td>19.12</td>
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<tr>
<td>2nd</td>
<td>$1.68 \times 10^8$</td>
<td>$7.96 \times 10^7$</td>
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<tr>
<td>3rd</td>
<td>$2.91 \times 10^8$</td>
<td>$2.86 \times 10^6$</td>
<td>28.48</td>
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<tr>
<td>4th</td>
<td>$4.33 \times 10^8$</td>
<td>$2.49 \times 10^7$</td>
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<td>5th</td>
<td>$6.72 \times 10^8$</td>
<td>$7.62 \times 10^6$</td>
<td>47.43</td>
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Table 1: Computation effort and performance for $\gamma = 1.6$ in example 1.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$|T_{zw}|_2$</th>
<th>Baeyens and Khargonekar (1994)</th>
<th>LMI toolbox</th>
<th>proposed optimization</th>
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<tr>
<td>1.6</td>
<td>28.13</td>
<td>37.20</td>
<td>22.88</td>
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<td>2.0</td>
<td>5.49</td>
<td>7.45</td>
<td>4.53</td>
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Table 2: Closed-loop performance of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controllers in example 1.
<table>
<thead>
<tr>
<th>$Q$ order</th>
<th>Flops</th>
<th>Computation Time (sec.)</th>
<th>$|T_{xz}|_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>std</td>
<td>mean</td>
</tr>
<tr>
<td>$1^{st}$</td>
<td>$5.08 \times 10^9$</td>
<td>$1.40 \times 10^9$</td>
<td>265.51</td>
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<tr>
<td>$2^{nd}$</td>
<td>$6.66 \times 10^9$</td>
<td>$3.23 \times 10^9$</td>
<td>307.77</td>
</tr>
<tr>
<td>$3^{rd}$</td>
<td>$8.08 \times 10^8$</td>
<td>$2.23 \times 10^9$</td>
<td>334.72</td>
</tr>
</tbody>
</table>

Table 3: Computation performance for $\gamma = 2.6$ in example 2.
Sznaier and Blanchini (1994) proposed optimization of $\gamma$th order $Q$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$|T_{z(t)}|_1$</th>
<th>$|T_{z(t)}|_1$</th>
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<tbody>
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<td>2.6</td>
<td>4.82</td>
<td>4.04</td>
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Table 4: Closed-loop performance of the mixed $L_1/H_\infty$ controllers in example 2.
Captions to Figures

Figure 1  LFT representation.
Figure 2  Multiobjective LFT form.
Figure 3  Computation effort (flops) as a function of the Q order for example 1 and $\gamma = 1.6$.
Figure 4  Evolution of cost function during the first stage of the optimization: example 1 and $\gamma = 2.0$.
Figure 5  Computation effort (flops) as a function of the Q order for example 2 and $\gamma = 2.6$.
Figure 6  Evolution of cost function during the first stage of the optimization: example 2.
Figure 1: LFT representation.
Figure 2: Multiobjective LFT form.
Figure 3: Computation effort (flops) as a function of the Q order for example 1 and $\gamma = 1.6$. 
Figure 4: Evolution of cost function during the first stage of the optimization: example 1 and $\gamma = 2.0$. 
Figure 5: Computation effort (flops) as a function of the Q order for example 2 and $\gamma = 2.6$. 
Figure 6: Evolution of cost function during the first stage of the optimization: example 2.