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Calculus of variations and field theory on graded fiber bundles

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Abstract

It is shown how to study higher order variational problems in graded fiber bundles through the Poincaré-Cartan form.

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1. Introduction

These notes deal with the generalization to supermanifolds of the classical approach to dynamical systems due to E. Cartan, by using the language of the calculus of variations on jet bundles. A basic object in this formalism is the so-called Poincaré$-$Cartan form, which can be defined in the non-graded setting through different constructions. Here, we simply recall that if we have a submersion $p : M \times \mathbb{R} \to \mathbb{R}$ with M an $m-$dimensional $\mathcal{C}^\infty$ manifold, on the tangent bundle we can construct a $(1, 1)$ tensor field on $J^1(p)$ (the space of $1-$jets of sections of p) through the formula $\mathcal{J} = J - \Delta \otimes dt$ (with the canonical almost tangent structure J and the Liouville vector field Δ) and, given a Lagrangian $L \in \mathcal{C}^\infty(J^1(p))$, the Poincaré−Cartan 1−form also on $J^1(p)$:

$$\Theta^L = \mathcal{L}_{\mathcal{J}} L + \lambda_L, \tag{1}$$

$\lambda_L = L \cdot p^*(dt)$ being the Lagrangian density associated to L, usually written simply as $\lambda_L = L \cdot dt$.

From this point on, a number of results can be deduced in a straightforward manner, such as the Euler−Lagrange equations or the Noether theorems. Our aim is to study first order Berezinian variational problems for fields on supermanifolds, but with the techniques of the graded calculus of variations, as these are well developed, see [1]. There exists a tool for doing this, known as the Comparison Theorem, but there is a price we must pay: the theorem guarantees that a first order Berezinian variational problem is equivalent (in a sense to be made precise later) to a graded variational problem, *but of higher order.*

2. Higher order Poincaré−Cartan superforms

For notations and basic results on supermanifolds, see [2], [5] and references therein.

We will work with a graded submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$, with M an oriented manifold with volume form $\eta = d\tilde{x}^1 \wedge ... \wedge d\tilde{x}^m = d^m\tilde{x}$. Negative indices denote odd coordinates, and Greek ones sum over even *and* odd indices. Often, use will be made of the notation $\eta^G = d^G x^1 \wedge ... \wedge d^G x^m = d^G_m x$ and $d^G_m \hat{x}^i$ will mean "$d^G_m x$ with the factor $d^G x^i$ (with $i \in \{1, ..., m\}$) omitted". Also, we will use the notation $(J^k_G(p), \mathcal{A}_{J^k_G(p)})$ (or sometimes $(J^k_G, \mathcal{A}_{J^k_G})$ if p is understood) for the graded bundle of k−jets over p. For definitions and details, see [1]. Fibered coordinates are chosen in such a way that, along a section σ of p, $j^k(\sigma)^*(y^\mu_\alpha) = \frac{\partial}{\partial x^\alpha}\sigma^*(y^\mu)$. Finally, L will denote an element of $\mathcal{A}_{J^1_G(p)}$, i.e, $L = L(x^\alpha, y^\mu, y^\mu_\alpha)$.

Now let ξ_L be a first order Berezinian density, which once a volume form on the base manifold η is chosen, can be written as $\xi = [d^G x^1 \wedge ... \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ ... \circ \frac{d}{dx^{-n}}]L$. Our intention is to study the Cartan formalism for variational problems, and in this formalism a central object is the so-called Cartan form, denoted Θ^L_0 and locally given by

$$\Theta^L_0 = \sum_{i=1}^{m} \sum_{\mu=-s}^{r} (-1)^{m+i} [d^G_m \hat{x}^i \wedge (d^G y^\mu - \sum_{\alpha=-n}^{m} d^G x^\alpha \cdot y^\mu_\alpha)] \frac{\partial L}{\partial y^\mu_i} + d^G_m x \cdot L.$$

The graded analog of the vertical endomorphism of the tangent bundle used in classical mechanics is,

$$J_1 := \sum_{i=1}^{m} \sum_{\beta=-s}^{r} (-1)^{m+i} d_m^G \hat{x}^i \wedge d^G y^\mu \otimes \frac{\partial}{\partial y_i^\mu},$$

also, for each $\alpha \in \{-n, ..., -1, 1, ..., m\}, i \in \{1, ..., m\}$, the graded analog of the Liouville vector field reads,

$$\Delta_{\alpha i} = \sum_{\beta=-s}^{r} (-1)^{m+i} y_\alpha^\mu \frac{\partial}{\partial y_i^\mu};$$

and finally (we omit the summation symbols from now on, by using Einstein's convention), let us define$\mathcal{J}_1 := J_1 - d_m^G \hat{x}^i \wedge d^G x^\alpha \otimes \Delta_{\alpha i}$.

If we evaluate $\mathcal{L}_{\mathcal{J}_1}^G(L)$, we obtain: $\mathcal{L}_{\mathcal{J}_1}^G(L) = \Theta_0^L - d_m^G x \cdot L$, so that

$$\Theta_0^L = \mathcal{L}_{\mathcal{J}_1}^G(L) + \eta^G \cdot L. \tag{2}$$

Formally, this expression is the same as (1), but now η^G appears and $\mathcal{J}_1$ also contains a factor of this kind. To have Θ_0^L intrinsically defined, it remains to prove that $\mathcal{J}_1$ is also intrinsically defined. Again, $\mathcal{J}_1$ is the graded analogue of the $(1, m)-$tensor field S_η that appears in the non-graded case, see [7] pgs $156 - 158$. We will study the intrinsic construction of these objects, but the generalization is not straightforward, as pointwise constructions are not applicable in a graded context.

The geometrical setting needed to make these ideas precise, rests upon the following results. Let $(N, \mathcal{B}) \xrightarrow{p} (M, \mathcal{A})$ be a graded submersion. Consider the cotangent supervector bundle on $(M, \mathcal{A})$, $\mathcal{ST}^*(M, \mathcal{A})$, and its pull-back to a supervector bundle on $(N, \mathcal{B})$, $p^*\mathcal{ST}^*(M, \mathcal{A})$. Also, let $\mathcal{V}(p) \subset \mathcal{ST}(N, \mathcal{B})$ be the vertical subspace of p, which is another supervector bundle on $(N, \mathcal{B})$. Thus, we have the product supervector bundle $p^*\mathcal{ST}^*(M, \mathcal{A}) \otimes \mathcal{V}(p)$ over $(N, \mathcal{B})$, with a projection which we will denote π, and we can construct the diagram

$$\begin{array}{ccccc} & & J_G^1(p) & & \\ & & \downarrow p_{1,0} & & \\ p^*\mathcal{ST}^*(M, \mathcal{A}) \otimes \mathcal{V}(p) & \xrightarrow{\pi} & J_G^0(p) \simeq (N, \mathcal{B}) & \xrightarrow{p} & (M, \mathcal{A}). \end{array}$$

Now, consider the submersion $p_{k-1} : J_G^{k-1}(p) \to (M, \mathcal{A})$ playing the rôle of p. Then, if $\{x^\alpha, y^\mu, z_I^\mu\}_{1 \leq |I| \leq k-1}$ is a system of coordinates for $J_G^{k-1}(p)$, $\{x^\alpha, y^\mu, z_I^\mu, w_K^\mu\}$ (with $1 \leq |K| \leq k$, $1 \leq |I| \leq k-1$ where the usual notation for multi-indices is employed) is a system for $J_G^1(p_{k-1})$ (note that, along sections,

$z_I^\mu = y_I^\mu$ and $w_{I+\alpha}^\mu = z_{\alpha I}^\mu$), and we have a family of graded $m-$forms with values on $\mathcal{V}((p_{k-1,1})_{1,0})$ whose local expressions are

$$\mathcal{J}_k = (-1)^{m-1}\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge \left(\theta^{y^\mu} \otimes \frac{\partial}{\partial y_i^\mu} + \theta^{z_I^\mu} \otimes \frac{\partial}{\partial z_{iI}^\mu} + \theta^{w_K^\mu} \otimes \frac{\partial}{\partial w_{iK}^\mu}\right)$$

(note that the sum $\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge \theta^{w_K^\mu} \otimes \frac{\partial}{\partial w_{iK}^\mu}$ runs only up to $|K| = k-1$), where $\theta^{z_I^\mu} = d^G z_I^\mu - d^G x^\alpha \cdot z_{\alpha I}^\mu$ and so on are the contact forms. The following theorem tells us that this construction can be made intrinsically.

Theorem 1 ([5]) *On $J_G^k(p)$ (for any k) there is defined a canonical graded (Poincaré-Cartan) $m-$form with values on $\mathcal{V}((p_k)_{1,0}) \subset \mathcal{V}((p_{k-1,1})_{1,0})$, which we will denote $\mathcal{J}_k$, and whose local expression is (writing collectively θ_I^μ instead of $\theta^{z_I^\mu}$ and $\theta^{w_K^\mu}$)*

$$\mathcal{J}_k = (-1)^{m-1}\iota_{\frac{\partial}{\partial x^i}}\eta^G \wedge \theta_I^\mu \otimes \frac{\partial}{\partial y_{I+i}^\mu},$$

where $0 \leq |I| \leq k-1$, with the usual convention $\theta_I^\mu = \theta^\mu$ when $|I| = 0$.

Definition 1 *Generalizing (2), we define, for any $L \in \mathcal{A}_{J_G^k}$, the graded $m-$form*

$$\tilde{\Theta}_k^L = \mathcal{L}_{J_k}^G(L) + \eta^G \cdot L. \tag{3}$$

These are the Poincaré−Cartan forms for higher order graded variational problems. In the next Section we will see how to use them in order to solve first order Berezinian variational problems.

3. The Comparison Theorem and Euler−Lagrange equations

Roughly speaking, the Comparison Theorem states the following (see [3]): given $\xi_L = \left[d^G x^1 \wedge ... \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \cdots \circ \frac{d}{dx^{-n}}\right] \cdot L$ (with $L \in \mathcal{A}_{J_G^1}$) a *first order* Berezinian density, the set of extremals for the variational problem it determines is in a bijective correspondence with the set of extremals for the variational problem associated to the $(n+1)-th$ *order* graded density $-\lambda_{\xi_L} = d^G x^1 \wedge ... \wedge d^G x^m \cdot \frac{d^n L}{dx^{-1} \cdots dx^{-n}}$ (where n is the odd dimension of $(M, \mathcal{A})$).

If we compare (2) and (3), we see that there are two a priori independent ways for getting the Poincaré−Cartan form for the densityλ_{ξ_L} . From (2), we could construct $\Theta_0^L = \mathcal{L}_{\mathcal{J}_1}^G(L) + \eta^G \cdot L$ and then compute

$$\Theta_{n+1}^L = \mathcal{L}^G_{\frac{d}{dx^{-1}}} \circ \cdots \circ \mathcal{L}^G_{\frac{d}{dx^{-n}}} \Theta_0^L, \tag{4}$$

or we could just apply the definition (3) to $\frac{d^n L}{dx^{-1}\cdots dx^{-n}} \in \mathcal{A}_{J_G^{n+1}}$ to get directly (let us denote $\tilde{\Theta}_k^{\frac{d^n L}{dx^{-1}\cdots dx^{-n}}}$ simply as $\tilde{\Theta}_k^L$ for obvious reasons)

$$\tilde{\Theta}_{n+1}^L = \mathcal{L}_{J_{n+1}}^G\left(\frac{d^n L}{dx^{-1}\cdots dx^{-n}}\right) + \eta^G \cdot \frac{d^n L}{dx^{-1}\cdots dx^{-n}}.$$

One of the main results in [5] is that these ways coincide.

Theorem 2 ([5]) *Let ξ_L be a first order Berezinian density, and let $\lambda_{\xi_L} = d^G x^1 \wedge \ldots \wedge d^G x^m \frac{d^n L}{dx^{-1}\ldots dx^{-n}}$. Let Θ_0^L be the graded Poincaré-Cartan form corresponding to $-\lambda_{\xi_L}$; then (with the notation as above),*

$$\Theta_{n+1}^L = \tilde{\Theta}_{n+1}^L.$$

Theorem 3 ([5]) *A local section s of $(N,\mathcal{B}) \xrightarrow{p} (M,\mathcal{A})$ is a critical section for the Berezinian density $\xi_L = [d^G x^1 \wedge \ldots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \ldots \circ \frac{d}{dx^{-n}}] \cdot L$ with $L \in \mathcal{A}_{J_G^1}$, if and only if it verifies*

$$(j^{n+1}s)^* \left(\iota_X d^G \Theta_{n+1}^L\right) = 0 \tag{5}$$

for every vector field X on $J_G^{n+1}(p)$, vertical over $(M,\mathcal{A})$.

Of course, the local version of (5) are the Euler−Lagrange equations:

$$(j^{n+1}s)^* \left(\frac{\partial L}{\partial y^\mu} - \frac{d}{dx^i}\frac{\partial L}{\partial y_i^\mu} - (-1)^\mu \frac{d}{dx^{-j}}\frac{\partial L}{\partial y_{-j}^\mu}\right) = 0.$$

4. Noether Theorem and supersymmetries

As in the classical (non-graded case), we can study the invariance of variational densities, and the associated Noether−type theorems..

Definition 2 *A $p-$projectable vector field X on $(N,\mathcal{B})$ is said to be an infinitesimal supersymmetry of the Berezinian density $\xi_L = [d^G x^1 \wedge \ldots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \ldots \circ \frac{d}{dx^{-n}}] \cdot L$ with $L \in \mathcal{A}_{J_G^1}$, if $\mathcal{L}^G_{X_{(n+1)}}\xi_L = 0$, where $X_{(n+1)}$ is the $(n+1)-$jet extension of X by contact graded infinitesimal transformations and $\mathcal{L}^G$ is the graded Lie derivative.*

Definition 3 *A graded vector field X' on $(M,\mathcal{A})$ is said to have a graded divergence with respect to a graded volume m-form η^G on $(M,\mathcal{A})$ if there exists a function $f \in \mathcal{A}$ such that $\mathcal{L}^G_{X'}\eta^G = \eta^G f$. In this case, we put $f = \mathrm{div}_G(X')$. A graded vector field X on $(N,\mathcal{B})$ is said to have divergence if it is $p-$projectable and if its projection X' has divergence.*

Definition 4 *Let X' be a graded vector field on $(M, \mathcal{A})$. Given a Berezinian density ξ on $(M, \mathcal{A})$, a function $g \in \mathcal{A}$ exists such that $\mathcal{L}^G_{X'}[\xi] = (-1)^{|X'||\xi|}[\xi] \cdot g$; we put $g = \mathrm{div}_B(X')$ and call it the Berezinian divergence of X'..*

Theorem 4 ([5]) *Assume X is an infinitesimal supersymmetry of the Berezinian density $\xi_L = [d^G x^1 \wedge \ldots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \ldots \circ \frac{d}{dx^{-n}}] \cdot L$ (with $L \in \mathcal{A}_{J^1_G}$) such that*

1. *The projection X' of X onto $(M, \mathcal{A})$ has divergence with respect to $d^G x^1 \wedge \ldots \wedge d^G x^m$,*
2. $\mathrm{div}_B(X') = \mathrm{div}_G(X')$.

Then, for every critical section s of ξ_L we have

$$d^G[(j^{n+1}s)^*(\iota_{X_{(n+1)}}\Theta^L)] = 0.$$

The superfunctions $\iota_{X_{(n+1)}}\Theta^L$ appearing in the statement, are called Noether supercurrents. Analogously, the graded vector fields X satisfying the conditions of the Theorem (and, in general, those leading to Noether supercurrents, note that these conditions are sufficient, not necessary) are called Noether supersymmetries.

Corollary 1 ([5]) *Assume X is a $p-$vertical graded vector field which also is an infinitesimal supersymmetry of the Berezinian density $\xi_L = [d^G x^1 \wedge \ldots \wedge d^G x^m \otimes \frac{d}{dx^{-1}} \circ \ldots \circ \frac{d}{dx^{-n}}]L$ (with $L \in \mathcal{A}_{J^1_G}$). Then, for every critical section s of ξ_L we have*

$$d^G[(j^{n+1}s)^*(\iota_{X_{(n+1)}}\Theta^L)] = 0.$$

5. An example

Consider $(M, C^\infty(M))$. In this supermanifold, there are no negative index supercoordinates so we will denote them by $\{x^i\}_{i=1}^{n=\dim M}$. A classical regular Lagrangian $L \in C^\infty(J^1(\pi : \mathbb{R} \times M \to M))$ can be lifted to $J^1(p : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1})$ and then we can apply all the results of the previous sections. In particular, we can determine a space denoted $(\mathcal{S}, \mathcal{A}_S)$, which will be called the space of solutions, in which $d\Theta^L$ is a symplectic superform; this space of solutions is graded isomorphic to $(TM, \Omega(TM))$ endowed with the Koszul-Schouten form Ξ_{KS} (Theorem 14.5 in [4]). If in TM we take the classical canonical coordinates given by L, $\{x^i, p^i = \frac{\partial L}{\partial x^i}\}_{i=1}^n$, then $\{x^i, p^i, x^{-i}, p^{-i}\}_{i=1}^n$ is a supercoordinate system on $(TM, \Omega(TM))$ and the Koszul-Schouten form is

$$\Xi_{KS} = \mathrm{d}^G x^{-i}\, \mathrm{d}^G p^i + \mathrm{d}^G x^i\, \mathrm{d}^G p^{-i}.$$

Moreover, the superhamiltonian vector field corresponding to the superfunction $E_L = L^G_{\frac{d}{ds}}(L - \Delta L)$ (the superenergy associated to the Lagrangian, here s is the odd coordinate in $\mathbb{R}^{1|1}$) is

$$X_{E_L} = -\mathcal{L}_{X_H} - \mathrm{d},$$

where H is the classical Hamiltonian associated to L (i.e, $H = x^i_t p^i - L$) and X_H is the corresponding Hamiltonian vector field given by the classical symplectic structure induced by L.

In this context, we can prove the following result:

Theorem 5 ([6]) *The space of solutions of the graded variational problem determined by a classical Lagrangian $L \in C^\infty(J^1(\pi : M \times \mathbb{R} \to \mathbb{R}))$, has the structure of a Batalin-Vilkovisky algebra, and the superenergy $S = E_L$ is a solution of the associated Master Equation.*

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