Harmonicity in supermanifolds and sigma models

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Abstract. We show that given an odd metric G on a supermanifold $(M,\mathscr{A})$ and its associated Laplacian Δ, it is possible to interpret harmonic superfunctions (i.e., those $f \in \mathscr{A}$ such that $\Delta f = 0$) as solutions to a variational problem describing a supersymmetric sigma model.

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1. INTRODUCTION

In Physics, by a non linear sigma model it is understood the study of a field theory described by the so called Polyakov action (see [1, 2, 10]), which reads

$$J = \frac{T}{2}\int d^m x \sqrt{|g|} g^{\mu\nu} h_{ij} \partial_\mu \phi^i \partial_\nu \phi^j. \tag{1}$$

This expression assumes that we have a mapping $\phi : M \to N$ between manifolds M, called the source ($m-$dimensional, endowed with coordinates $\{x^\alpha\}_{\alpha=1}^m$ and a metric $g = g_{\mu\nu}(x)$), and N, called the target ($n-$dimensional, endowed with coordinates $\{y^i\}_{i=1}^n$ and a metric $h = h_{ij}(y)$). As an additional notation, $\phi^i = y^i \circ \phi : M \to \mathbb{R}$ is a set of components for ϕ, $|g| = |\det g|$ and $g^{\mu\nu}$ are the components of the inverse metric g^{-1}. This model is usually applied when $m = 2$, that is, the manifold M is a surface. In this case, the surface is viewed as the result of the time evolution of a $1-$dimensional object (a string) and T is interpreted as the tension of the string.

It is customary to write

$$\gamma_{\mu\nu} = \partial_\mu \phi^i \partial_\nu \phi^j h_{ij}$$

and to call it the "induced" metric on M (actually, this is just the pullback $\phi^* h$). It is particularly interesting, for physical reasons, to consider the situation in which the metrics available on M (the original g and the induced γ) are conformal, that is, there exists a function $f \in C^\infty(M)$ with $f(x) > 0$ for all $x \in M$ such that

$$g_{\mu\nu}(x) = f(x) \cdot \gamma_{\mu\nu}(x). \tag{2}$$

In this case, the Polyakov action reduces to

$$\int d^m x \sqrt{|g|} g^{\mu\nu} \gamma_{\mu\nu} = \int d^m x \sqrt{|\gamma|} f^{\frac{n-2}{2}}.$$

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The case $m = 2$ is very special, for on any surface all the metrics are conformally related, so condition (2) is always satisfied. Hence, the action for a bidimensional (source) sigma model is simply

$$S = \int d^2x \sqrt{|\gamma|},$$

which is known in Physics as the Nambu-Goto action for the bosonic string (in Mathematics, it is just the Riemannian area functional).

So, from a physicist's point of view, the interest of studying non linear sigma models as (1) is due to the fact that these encode the field equations of fields defined on extended objects that generalize (bosonic) strings (regarding terminology, we should say that when we speak about *bosonic* strings it is understood that the target N is a (pseudo)Riemannian space. For *fermionic* strings or *superstrings*, N is a supermanifold).

A question of the upmost importance in Physics, is that of the invariance properties of a model. In our case, it is easy to see that non linear sigma models (as described by the Polyakov action) have the following features:

1. Invariance under the diffeomorphism groups $\mathscr{D}iff(M)$ and $\mathscr{D}iff(N)$.
2. Invariance under $Iso(M)$, the group of isometries of M (a particular case of (1)).
3. If $n = 2$, invariance under $Wey(N,h)$, the conformal transformations of (N,h).
4. If (N,h) is taken to be the Minkowski spacetime, invariance under the Poincaré group $\mathscr{P}$.

All these properties become apparent if we write Polyakov's action in an intrinsic manner. It is easy to see that (1) is just the local expression of what in differential geometry is called the energy of a map $\phi : M \to N$ between (pseudo)Riemannian manifolds (see [4, 3, 11]). The energy functional is

$$E = \frac{1}{2}\int_M Tr(g^{-1}(\phi^* h))\, dvol_g. \tag{3}$$

The classical interpretation of this functional is the following. If we have a smooth map $\phi : (M,g) \to (N,h)$ between Riemannian manifolds and we imagine them made of rubber and marble, respectively, then applying M over N requires a certain amount of energy (work), which is stored in the tension of the surface.

2. HARMONIC FUNCTIONS AND LAPLACIANS

Critical points of sigma models (1) are the same as critical points of the energy functional (3). They only differ in the name they receive from physicists and mathematicians, the latter preferring the name *harmonic mappings* for these critical points.

Harmonic mappings encompass a lot of situations of geometric interest, we stress the following:

1. If $(M,g) = (\mathbb{R}, g_{eu})$, the Euclidean line, harmonic mappings $\phi : \mathbb{R} \to (N,h)$ are the geodesics of (N,h).
2. If $(N,h) = (\mathbb{R}, g_{eu})$, then harmonic mappings $\phi : (M,g) \to \mathbb{R}$ are the harmonic functions on (N,h).

There are other cases: Minimal immersions, totally geodesic maps, holomorphic maps between Kähler manifolds, etc (see [9], Sect. 3.4.) But let us focus our attention in the case (2), that of harmonic functions.

What are the extremals of (3) when $(N,h) = (\mathbb{R}, g_{eu})$? How do we know that these are precisely the harmonic functions? In a coordinate system, the Euler-Lagrange equations for (3) are

$$\Delta\phi \equiv \frac{1}{\sqrt{\det g}}\,\partial_\mu(\sqrt{\det g}\,g^{\mu\nu}\partial_\nu\phi) = 0, \tag{4}$$

where Δ is the classical Laplace operator defined by g (see [11, 9]). This can also be written as

$$\Delta\phi \equiv (\mathrm{div}\circ\mathrm{grad})\phi = 0, \tag{5}$$

and our goal is precisely to study the translation of this setting to the context of supermanifolds. That is: we want to write down the solutions of a variational characterization of supersigma models as harmonic superfunctions.

Let us make a side comment. There is another way of expressing these equations, as

$$\Delta\phi \equiv (\delta d + d\delta)\phi = 0,$$

where $\delta = -*d*$ is the codifferential associated to g through the Hodge star $*$. This equation is the starting point of Hodge theory, but it presents a very difficult problem of interpretation in the context of supermanifolds as for these there is no top on cohomology, so we can not define in a direct way Hodge duality and the associated calculus. This is the reason for choosing (5) as the intrinsic expression for the Laplacian.

Another comment refers to the kind of Riemannian metrics we will use (odd metrics). The even and odd cases have quite different behaviours, and our choice here is determined by the fact that we will deal with superfunctions, that is, morphisms of supermanifolds where the target is $\mathbb{R}^{1|1}$. For this supermanifold, there exist no (non degenerate) even metrics, as this requires an even-dimensional base manifold (see [7]). Thus, as morphisms of supermanifolds must be even, we are forced to work with odd metrics in the source manifold (cf. Equation (6) below).

3. SUPERSIGMA MODELS AND HARMONIC SUPERFUNCTIONS

There is a problem that we must overcome to achieve our goal: For a supermanifold, a single natural definition of integral does not exist. There are two constructions, the Berezin integral and the graded integral, very different in nature. The Berezin integral is the one is used by physicists, while the graded integral has better mathematical properties. Berezin's integral does not really integrate forms on supermanifolds, but sections of the so called Berezinian module. On the other hand, the graded integral has an associated calculus of variations, actually integrates superforms... But it does not provide the equations physicists want. In order to surpass these difficulties, we make use of the Comparison Theorem (see [6, 5]) which relates solutions of a given Berezinian variational problem with the corresponding ones of a certain associated graded variational problem of higher order. In [8], we have applied this technique to the case of the Polyakov functional (energy functional) for supermanifolds $(M,\mathscr{A})$ and $(N,\mathscr{B})$ with a morphism $\phi:(M,\mathscr{A})\to(N,\mathscr{B})$ when $(N,\mathscr{B})=\mathbb{R}^{1|1}$, that is, we have studied the variational characterization of harmonic superfunctions (or $(1|1)-$linear supersigma models).

Given the morphism $\phi:(M,\mathscr{A})\to(N,\mathscr{B})$ and odd supermetrics G, on $(M,\mathscr{A})$, and H, on $(N,\mathscr{B})$, we define a supersigma model by the action functional

$$J=\int_{\mathrm{Ber}\,\xi_G} C_1^1(G^{-1}(\phi^* H)), \tag{6}$$

where ξ_G is the canonical section of the Berezinian module given by the Berezinian Riemannian section associated to G and $\int_{\mathrm{Ber}}$ denotes the Berezin integral. Notice that we do not use the supertrace, but the common contraction of supertensors (for H the canonical odd supermetric on $\mathbb{R}^{1|1}$ the supertrace would give an identically zero expression). With the aid of the Comparison Theorem, we can study the extremals of this functional through an associated graded variational problem for which we have a calculus of variations available. Indeed, the computations show that these extremals are the same obtained in Proposition 3 below.

In the following we will assume that $(M,\mathscr{A})$ is a split supermanifold, that is, $\mathscr{A}=\Gamma(\bigwedge E)$ where $E\xrightarrow{\pi}M$ is a vector bundle over M. We write $\{x^i,x^{-j}\}$ $(1\le i\le m=dimM$, $1\le j\le n)$ for a supercoordinate system on $(M,\mathscr{A})$.

We can summarize our results about the problem of relating extremals of the Polyakov (or energy) functional for supermanifolds and harmonic superfunctions by saying that the main relations and properties of the classical case also holds for supermanifolds:

1. We can define, for superfunctions $f\in\mathscr{A}=\Gamma(\bigwedge E)$, a notion of gradient with respect to an odd supermetric G on $(M,\mathscr{A})$:

   Proposition 1: *For a superfunction f, the following local expression holds true:*

   $$\mathrm{grad}_G f = g^{ij}\frac{\partial f}{\partial x^{-j}}\frac{\partial}{\partial x^i}+g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^{-l}},$$

   g^{ij} being the components of the inverse metric g^{-1} of g on M.
2. For a vector field on $(M,\Gamma(\bigwedge E))$ (that is, a derivation of $\Gamma(\bigwedge E)$), such as $\mathrm{grad}_G f$, we can define and compute its Berezinian divergence with respect to the Riemannian Berezinian section ξ_G, denoted div_{ξ_G}:

   Proposition 2: *For any $D\in Der\Gamma(\bigwedge E)$, we have*

   $$\mathrm{div}_{\xi_G}(D)=(-1)^{|D||\xi_G|}\frac{1}{|G|}\frac{\partial}{\partial x^\alpha}(|G|\cdot D^\alpha),$$

   where $|\xi_G|$ is the degree of the Riemannian Berezinian section, $|G|$ is the Berezin determinant of G, and $\alpha=\{i,-j\}$ denotes even and odd indices collectively.
3. We can define the superLaplacian on superfunctions as $\Delta=\mathrm{div}_{\xi_G}\circ\mathrm{grad}_G$ and study its kernel:

Proposition 3: *For any superfunction $f \in \Gamma(\bigwedge E)$, the equation $\Delta f = 0$ is equivalent to*

$$\frac{1}{|G|}\frac{\partial |G|}{\partial x^i} g^{ij} f_{-j} + \frac{\partial g^{ij}}{\partial x^i} f_{-j} + g^{ij} f_{i,-j} + g^{ij} f_{-i,j} = 0.$$

4. And, finally, we can relate these equations to the ones resulting from the computation of the extremals of (6):

   Theorem: *The harmonic superfunctions (the $f \in \Gamma(\bigwedge E)$ such that $\Delta f = 0$) are precisely the solutions of the Euler-Lagrange equations of the $(1|1)-$linear supersymmetric sigma model.*

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