Ashdin Publishing
Journal of Generalized Lie Theory and Applications
Vol. 5 (2011), Article ID G100801, 10 pages
doi:10.4303/jglta/G100801

Research Article

An Approach to Omni-Lie Algebroids Using Quasi-Derivations

Dennise García-Beltrán and José A. Vallejo

Faculty of Sciences, State University of San Luis Potosi, Lat. Av. Salvador Nava s/n, 78290 San Luis Potosí, Mexico
Address correspondence to José A. Vallejo, jvallejo@fc.uaslp.mx

Received 4 August 2010; Accepted 10 January 2011

Abstract We introduce the notion of left (and right) quasi-Loday algebroids and a "universal space" for them, called a left (right) omni-Loday algebroid, in such a way that Lie algebroids, omni-Lie algebras and omni-Loday algebroids are particular substructures.

MSC 2010: 17A32, 53D17, 58H99

1 Introduction

There are several ways in which Lie algebras can be generalized. Recall that if M is an $\mathcal{R}$-module[1] endowed with an $\mathcal{R}$-bilinear bracket $[\ ,\ ]: M \times M \to M$ such that for all $u, v, w \in M$

(i) $[u, v] = -[v, u]$ (antisymmetry),
(ii) $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$ (Jacobi identity),

then $(M, [\ ,\ ])$ is a Lie algebra structure over M. When $\mathcal{R} = \mathbb{R}$ (resp. $\mathbb{C}$) we speak about a real Lie algebra (resp. a complex Lie algebra).

Let us briefly mention some of these generalizations.

(a) We can lift the restriction of antisymmetry. We then get the notion of Loday (or Leibniz[2]) algebra [6,8]. More precisely, the pair $(M, [\ ,\ ])$ is a left Loday algebra if, instead of conditions (i), (ii) above, it satisfies the left Leibniz identity:
  (iii) $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$.
  Note that this condition can be expressed by saying that $[u,\ ]$ is a derivation with respect to the product $[\ ,\ ]$. Analogously, we can define right Loday algebras over M, by imposing that $[\ , w]$ is a derivation with respect to $[\ ,\ ]$:
  (iii') $[[u, v], w] = [[u, w], v] + [u, [v, w]]$.
  Note that any one of (iii) or (iii') is equivalent to the Jacobi identity when $[\ ,\ ]$ is antisymmetric.
(b) Also, it is possible to consider a family of Lie algebras parameterized by points on a manifold M which, with some natural geometric assumptions, leads to the idea of Lie algebroid introduced by J. Pradines (see [5,9]). To be precise, a Lie algebroid over a manifold M (assume it real for simplicity) is given by a vector bundle $\pi: E \to M$, an $\mathbb{R}$-bilinear bracket $[\ ,\ ]: \mathit{\Gamma} E \times \mathit{\Gamma} E \to \mathit{\Gamma} E$ defined on the $\mathcal{C}^\infty(M)$-module of sections of E, and a mapping $q_E: \mathit{\Gamma} E \to \mathfrak{X}(M)$ (called the anchor map) such that, for all $X, Y \in \mathit{\Gamma} E$, $f \in \mathcal{C}^\infty(M)$:
  (1) $(\mathit{\Gamma} E, [\ ,\ ])$ is a real Lie algebra,
  (2) $[X, fY] = f[X, Y] + q_E(X)(f) \cdot Y$.
  Note that, in the case when M reduces to a single point, a Lie algebroid over $M = \{*\}$ is just a Lie algebra. A basic property of Lie algebroids is that the anchor map q_E is a Lie algebra morphism when the bracket on $\mathfrak{X}(M)$ is taken as the Lie bracket of vector fields [2,7].
(c) Finally, A. Weinstein introduced in [15] the concept of omni-Lie algebra, a structure that can be thought of as a kind of "universal space" for Lie algebras: take any natural number $n \geq 2$ and consider the space product $\mathcal{E}_n = \mathfrak{gl}_n \times \mathbb{R}^n$ endowed with the $\mathbb{R}$-bilinear form $\{\ ,\ \}: \mathcal{E}_n \times \mathcal{E}_n \to \mathcal{E}_n$ given by

$$\{(A, x), (B, y)\} = \left([A, B], \frac{1}{2}(Ay - Bx)\right),$$

[1] Here $\mathcal{R}$ is a commutative ring with unit element $1_{\mathcal{R}}$.
[2] We prefer the denomination "Loday algebra"; see [6] for an explanation.

where $[A, B] = A \circ B - B \circ A$ is the Lie bracket of $\mathfrak{gl}_n$. Then, $(\mathcal{E}_n, \{\ ,\ \})$ is the n-dimensional omni-Lie algebra. The reason behind this denomination is that any n-dimensional real Lie algebra $\mathfrak{g}$ is a closed maximal subspace of $(\mathcal{E}_n, \{\ ,\ \})$.

Our goal is to define a structure for which the constructions mentioned in (a), (b), (c) appear as particular cases. In an absolutely unimaginative way, we will call it a left omni-Loday algebroid (of course, there exists the corresponding "right" definition). As we will see, this also includes as a particular case the notion of omni-Lie algebroid. Actually, the object we will construct will carry on a bracket that has already appeared in the literature, although under a different approach. In [4], M. K. Kinyon and A. Weinstein attacked the problem of integrating (in the sense of S. Lie's "Third Theorem") a Loday algebra[3], and they gave the following example: take $(\mathfrak{h}, [\ ,\ ])$ a Lie algebra, and let V be an $\mathfrak{h}$-module with left action on V given by $(\zeta, x) \mapsto \zeta x$. Then, we have the induced left action of $\mathfrak{h}$ on $\mathfrak{h} \times V$:

$$\zeta(\eta, y) = \big([\zeta, \eta], \zeta y\big).$$

A binary operation $\cdot$ can be defined on $\mathcal{E} = \mathfrak{h} \times V$ through

$$(\zeta, x) \cdot (\eta, y) = \zeta(\eta, y) = \big([\zeta, \eta], \zeta y\big).$$

It turns out that $(\mathcal{E}, \cdot)$ is a Loday algebra, and if $\mathfrak{h}$ acts nontrivially on V, then $(\mathcal{E}, \cdot)$ is *not* a Lie algebra. Kinyon and Weinstein called $\mathcal{E}$ with this Loday algebra structure the hemisemidirect product of $\mathfrak{h}$ with V. Our omni-Loday algebroid will be a particular case of this construction, taking $\mathfrak{gl}(V)$ as $\mathfrak{h}$ (see Definition 17).

To achieve our goal, let us note that it is necessary to recast the definition of a Lie algebroid in a form more suitable to an algebraic treatment, as in (a), (c). This can be easily done, just note that $\mathcal{C}^\infty(M)$ can be replaced by any $\mathcal{R}$-algebra $\mathcal{A}$, with unit element $1_{\mathcal{A}}$, and commutative ΓE by a faithful $\mathcal{A}$-module $\mathcal{F}$, and $\mathfrak{X}(M)$ by the module of derivations $\mathrm{Der}_{\mathcal{R}}(\mathcal{A})$.

This idea was cleverly exploited by J. Grabowski [2] who used it to prove that the property of the anchor map is a Lie algebra morphism. In the same paper, it is proved that there exist obstructions to the existence of Loday algebroid structures on vector bundles over a manifold M, stated in terms of the rank of these bundles (see Theorems 11, 12 below). As we will see, we can bypass these obstructions by considering left and right structures separately.

2 Quasi-derivations

The basic properties of a Lie algebroid are encoded in its anchor map, which in this context is a mapping $\rho : \mathcal{F} \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A})$. We will assume that $\mathcal{F}$ is endowed with an $\mathcal{R}$-bilinear bracket $[\![\ ,\ ]\!]$, then ρ is determined by two adjoint maps $\mathrm{ad}^L_{\mathcal{A}}, \mathrm{ad}^R_{\mathcal{A}} : \mathcal{F} \to \mathrm{End}_{\mathcal{R}}(\mathcal{F})$, given respectively by $\mathrm{ad}^L_{\mathcal{A}}(X) = [\![X,\ ]\!]$, $\mathrm{ad}^R_{\mathcal{A}}(X) = [\![\ , X]\!]$.

Under certain mild conditions, these mappings are quasi-derivations of $\mathcal{F}$, a property which is basic in the study of ρ. For instance, the fact that $\mathrm{ad}^L_{\mathcal{A}}$, $\mathrm{ad}^R_{\mathcal{A}}$ are quasi-derivations allows us to prove that ρ is a morphism of Lie algebras (when $(\mathcal{F}, [\![\ ,\ ]\!])$ is Lie and we take the commutator of endomorphisms as the bracket on $\mathrm{End}_{\mathcal{R}}(\mathcal{F})$); see [2] (we refer the reader to that paper for the proof of the results stated in this section).

We recall that an operator $D \in \mathrm{End}_{\mathcal{R}}(\mathcal{F})$ is a quasi-derivation if for a given $f \in \mathcal{A}$ there exists $g \in \mathcal{A}$ such that

$$[D, \mu_f] = \mu_g,$$

where $[\ ,\ ]$ is the commutator of endomorphisms of $\mathcal{F}$, and $\mu_h(X) = h \cdot X$, for any $h \in \mathcal{A}$, $X \in \mathcal{F}$. A quasi-derivation is called a tensor operator when

$$[D, \mu_f] = 0, \quad \forall f \in \mathcal{A}.$$

Note that this is equivalent to D being $\mathcal{A}$-linear (and not just $\mathcal{R}$-linear). Some other straightforward properties of quasi-derivations are as follows:

(1) the set of all the quasi-derivations of $\mathcal{F}$, $\mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ is an $\mathcal{R}$-module;
(2) the commutator of endomorphisms on $\mathrm{Der}_{\mathcal{R}}(\mathcal{F})$ restricts to a closed bracket on $\mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ (i.e. if $D_1, D_2 \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, then $[D_1, D_2] \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$). Thus $(\mathrm{QDer}_{\mathcal{R}}(\mathcal{F}), [\ ,\ ])$ inherits the Lie algebra structure of $(\mathrm{End}_{\mathcal{R}}(\mathcal{F}), [\ ,\ ])$;

[3] For more recent results in this topic, called the coquecigrue problem, see [3, 10, 13, 12] and references therein.

(3) $\mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ is not just an $\mathcal{R}$-module. Defining, for any $f \in \mathcal{A}$ and $D \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$,

$$f \cdot D := \mu_f \circ D,$$

it results that $\mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ is an $\mathcal{A}$-module too;

(4) $(\mathrm{QDer}_{\mathcal{R}}(\mathcal{F}), [\ ,\ ])$ is not just a Lie algebra, but also a Poisson algebra (with the product given by the composition of endomorphisms).

The following results will be crucial in the sequel.

Theorem 1. *There exists an $\mathcal{R}$-linear mapping* $\widehat{\ }\colon \mathrm{QDer}_{\mathcal{R}}(\mathcal{F}) \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A})$ *such that*

$$[D, \mu_f] = \mu_{\widehat{D}(f)}.$$

Corollary 2. *The $\mathcal{R}$-linear mapping $\widehat{\ }$extends to a Lie algebra morphism:*

$$[\widehat{D_1}, \widehat{D_2}] = [\widehat{D_1, D_2}], \quad \forall D_1, D_2 \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F}).$$

Combining (4) above with Theorem 1, we also get the following corollary.

Corollary 3. *If $\mathcal{A}$ is an $\mathcal{R}$-commutative algebra, the commutator $[\ ,\ ]$ on $\mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ satisfies*

$$[D_1, f \cdot D_2] = f \cdot [D_1, D_2] + \widehat{D_1}(f) \cdot D_2,$$

for all $D_1, D_2 \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, $f \in \mathcal{A}$.

3 Left Loday quasi-algebroids

The formula obtained in Corollary 3 looks very similar to condition (b2) in the definition of Lie algebroid. We can formalize this observation generalizing at once the definition, simply by replacing the Lie structure on ΓE (our $\mathcal{F}$ in the algebraic setting) by a Loday one. Thus, let $(\mathcal{F}, [\![\ ,\ ]\!])$ be a left Loday algebra. Given an $X \in \mathcal{F}$, denote by $\mathrm{ad}^L_X, \mathrm{ad}^X_R : \mathcal{F} \to \mathcal{F}$ the endomorphisms $\mathrm{ad}^L_X(Y) = [\![X, Y]\!]$, $\mathrm{ad}^R_X(Y) = [\![Y, X]\!]$.

Note that if $[\![\ ,\ ]\!]$ is antisymmetric, then $\mathrm{ad}^L_X = \mathrm{ad}^R_{-X}$.

Definition 4. The pair $(\mathcal{F}, [\![\ ,\ ]\!])$ is called a left Loday quasi-algebroid if $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, for all $X \in \mathcal{F}$.

This amounts to the condition that, given $X \in \mathcal{F}$, $f \in \mathcal{A}$,

$$[\![X, f \cdot Y]\!] - f \cdot [\![X, Y]\!] = [\mathrm{ad}^L_X, \mu_f](Y) = \mu_{\widehat{\mathrm{ad}^L_X}(f)}(Y) = \widehat{\mathrm{ad}^L_X}(f) \cdot Y, \quad \forall Y \in \mathcal{F},$$

and motivates the following definition.

Definition 5. The mapping

$$q^L_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathrm{Der}_{\mathcal{R}}(\mathcal{A}), \quad X \longmapsto q^L_{\mathcal{F}}(X) := \widehat{\mathrm{ad}^L_X}$$

is called the anchor of the left Loday quasi-algebroid. If $q^L_{\mathcal{F}}$ is tensorial, it is said that $(\mathcal{F}, [\![\ ,\ ]\!], q^L_{\mathcal{F}})$ is a left Loday algebroid on $\mathcal{F}$.

The condition in Definition 4 now reads

$$[\![X, f \cdot Y]\!] = f \cdot [\![X, Y]\!] + q^L_{\mathcal{F}}(X)(f) \cdot Y,$$

with this justifying the terminology with the “left” prefix.

Remark 6. There is the corresponding notion of right Loday quasi-algebroid, when $\mathrm{ad}^R_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$. In this case, the formula reads

$$[\![f \cdot X, Y]\!] = f \cdot [\![X, Y]\!] + q^R_{\mathcal{F}}(Y)(f) \cdot X.$$

Theorem 7. *Let $(\mathcal{F}, [\![\ ,\ ]\!], q^L_{\mathcal{F}})$ be a left Loday quasi-algebroid. Then, $q_{\mathcal{F}} : (\mathcal{F}, [\![\ ,\ ]\!]) \to (\mathrm{Der}_{\mathcal{R}}(\mathcal{A}), [\ ,\ ])$ is a morphism of left Loday $\mathcal{R}$-algebras.*

Proof. First, let us note that the condition of $[\![\ ,\ ]\!]$ being a Loday bracket on $\mathcal{F}$ means that

$$\left[\mathrm{ad}^L_X, \mathrm{ad}^L_Y\right] = \mathrm{ad}^L_{[\![X,Y]\!]} \quad \forall X, Y \in \mathcal{F}.$$

To check this, let $Z \in \mathcal{F}$ and compute

$$\begin{aligned}\left[\mathrm{ad}^L_X, \mathrm{ad}^L_Y\right](Z) &= \mathrm{ad}^L_X\left(\mathrm{ad}^L_Y(Z)\right) - \mathrm{ad}^L_Y\left(\mathrm{ad}^L_X(Z)\right) = [\![X,[\![Y,Z]\!]]\!] - [\![Y,[\![X,Z]\!]]\!] \\ &= [\![[\![X,Y]\!],Z]\!] + [\![Y,[\![X,Z]\!]]\!] - [\![Y,[\![X,Z]\!]]\!] = [\![[\![X,Y]\!],Z]\!] \\ &= \mathrm{ad}^L_{[\![X,Y]\!]}(Z).\end{aligned}$$

As this is valid for all $Z \in \mathcal{F}$, we get the stated equivalence.

Now, Corollary 2 says that for all $X, Y \in \mathcal{F}$,

$$\left[q^L_{\mathcal{F}}(X), q^L_{\mathcal{F}}(Y)\right] = \left[\widehat{\mathrm{ad}^L_X}, \widehat{\mathrm{ad}^L_Y}\right] = \widehat{\left[\mathrm{ad}^L_X, \mathrm{ad}^L_Y\right]} = \widehat{\mathrm{ad}^L_{[\![X,Y]\!]}} = q^L_{\mathcal{F}}\left([\![X,Y]\!]\right).$$ □

Remark 8. This result partly answers a question raised in [14, Remark 3.3 (1)].

The definitions just given can be particularized to the case of Lie algebras (i.e. $[\![\ ,\ ]\!]$ antisymmetric).

Definition 9. Let $(\mathcal{F}, [\![\ ,\ ]\!])$ be a Lie algebra. If $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, for all $X \in \mathcal{F}$, we say that $(\mathcal{F}, [\![\ ,\ ]\!], q_{\mathcal{F}})$, is a Lie quasi-algebroid, where

$$q_{\mathcal{F}} : F \longrightarrow \mathrm{Der}_{\mathcal{R}}(\mathcal{A}), \quad X \longmapsto q_{\mathcal{F}}(X) := \widehat{\mathrm{ad}^L_X}$$

is the anchor map. If $q_{\mathcal{F}}$ is tensorial ($\mathcal{A}$-linear), then we say that $(\mathcal{F}, [\![\ ,\ ]\!], q_{\mathcal{F}})$ is a Lie algebroid.

Remark 10. Note that in this case the distinction between the left and right cases is irrelevant: each left Lie quasi-algebroid with anchor $q_{\mathcal{F}}$ is also a right Lie quasi-algebroid with anchor $-q_{\mathcal{F}}$.

How different are left (and right) Loday quasi-algebroids, Lie quasi-algebroids and Loday algebroids? In some cases, there is no such distinction: if we take $\mathcal{R} = \mathbb{R}$, $\mathcal{A} = \mathcal{C}^\infty(M)$, $\mathcal{F} = \mathit{\Gamma} E$, with $\pi : E \to M$ a vector bundle over a manifold M, Grabowski calls a QD-Loday (resp. Lie) algebroid a left Loday (resp. Lie) quasi-algebroid (i.e. $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, for all $X \in \mathcal{F}$) such that $\mathrm{ad}^R_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, for all $X \in \mathcal{F}$; then, he proves the following.

Theorem 11. *Every QD-Loday algebroid (resp. QD-Lie) with $rank \geq 1$ is a Loday algebroid (resp. Lie).*

Theorem 12. *Every QD-Loday algebroid of rank 1, is a QD-Lie algebroid.*

4 Generation of Loday algebroids

As we have seen in the previous section, in order to get genuine examples of Loday quasi-algebroids, we must avoid that the two conditions $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ and $\mathrm{ad}^R_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$ are satisfied *simultaneously*. To get examples of this situation, it is useful to know how to generate Loday brackets from operators with certain features. First of all, note that given a left Loday bracket $[\![\ ,\ ]\!] : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, if we define

$$[\![\ ,\ ]\!]' : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}, \quad (X,Y) \longmapsto [\![X,Y]\!]' := [\![Y,X]\!],$$

then we have that $[\![\ ,\ ]\!]'$ is $\mathcal{R}$-bilinear and for all $X, Y, Z \in \mathcal{F}$,

$$\begin{aligned}[\![X,[\![Y,Z]\!]']\!]' + [\![[\![X,Z]\!]',Y]\!]' &= [\![X,[\![Z,Y]\!]]\!]' + [\![[\![Z,X]\!],Y]\!]' = [\![[\![Z,Y]\!],X]\!] + [\![Y,[\![Z,X]\!]]\!] \\ &= [\![Z,[\![Y,X]\!]]\!] = [\![[\![X,Y]\!]',Z]\!]',\end{aligned}$$

thus, $[\![\ ,\ ]\!]'$ is a right Loday bracket.

Analogously, given a right Loday bracket we can define a left Loday one, obtaining a correspondence between left and right Loday algebras.

Proposition 13. *If $(\mathcal{A}, \cdot)$ is an associative $\mathcal{R}$-algebra*[4] *and, moreover, is endowed with an $\mathcal{R}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ verifying*

$$D\big(a \cdot \big(D(b)\big)\big) = D(a) \cdot D(b) = D\big(\big(D(a)\big) \cdot b\big), \quad \forall a, b \in \mathcal{A},$$

[4] That is, $\mathcal{A}$ is an $\mathcal{R}$-module endowed with an associative product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

then one can define

$$[\ ,\ ]:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A},\quad (a,b)\longmapsto[a,b]:=D(a)\cdot b-b\cdot D(a),$$

which satisfies the properties of $\mathcal{R}$-bilinearity and the left Leibniz rule (so, it is a left Loday algebra).

Proof. Let us check first the $\mathcal{R}$-bilinearity

$$\begin{aligned}[][\alpha a+\beta b,c]&=D(\alpha a+\beta b)\cdot c-c\cdot D(\alpha a+\beta b)\\&=\alpha D(a)\cdot c+\beta D(b)\cdot c-\alpha c\cdot D(a)-\beta c\cdot D(b)\\&=\alpha\big(D(a)\cdot c-c\cdot D(a)\big)+\beta\big(D(b)\cdot c-c\cdot D(b)\big)\\&=\alpha[a,c]+\beta[b,c]\\ [a,\beta b+\gamma c]&=D(a)\cdot(\beta b+\gamma c)-(\beta b+\gamma c)\cdot D(a)\\&=\beta D(a)\cdot b+\gamma D(a)\cdot c-\beta b\cdot D(a)-\gamma c\cdot D(a)\\&=\beta\big(D(a)\cdot b-b\cdot D(a)\big)+\gamma\big(D(a)\cdot c-c\cdot D(a)\big)\\&=\beta[a,b]+\gamma[a,c].\end{aligned}$$

For the left Leibniz rule, we have

$$\begin{aligned}\big[a,[b,c]\big]-\big[[a,b],c\big]-\big[b,[a,c]\big]&=D(a)\cdot[b,c]-[b,c]\cdot D(a)-D\big([a,b]\big)\cdot c+c\cdot D[a,b]\\&\quad-D(b)\cdot[a,c]+[a,c]\cdot D(b)\\&=D(a)\cdot\big(D(b)\cdot c-c\cdot D(b)\big)-\big(D(b)\cdot c-c\cdot D(b)\big)\cdot D(a)\\&\quad-D\big(D(a)\cdot b-b\cdot D(a)\big)\cdot c+c\cdot D\big(D(a)\cdot b-b\cdot D(a)\big)\\&\quad-D(b)\cdot\big(D(a)\cdot c-c\cdot D(a)\big)+\big(D(a)\cdot c-c\cdot D(a)\big)\cdot D(b)\\&=D(a)\cdot D(b)\cdot c-D(a)\cdot c\cdot D(b)-D(b)\cdot c\cdot D(a)\\&\quad+c\cdot D(b)\cdot D(a)-D(a)\cdot D(b)\cdot c+D(b)\cdot D(a)\cdot c\\&\quad+c\cdot D(a)\cdot D(b)-c\cdot D(b)\cdot D(a)-D(b)\cdot D(a)\cdot c\\&\quad+D(b)\cdot c\cdot D(a)+D(a)\cdot c\cdot D(b)-c\cdot D(a)\cdot D(b)\\&=0.\end{aligned}$$

□

Example 14. Some examples of such mappings $D:\mathcal{A}\to\mathcal{A}$ are the following.

(a) The identity $D=Id$. In this particular case we obtain a Lie algebra.
(b) A zero-square derivation D. Indeed, if this is the case,

$$D\big(a\cdot D(b)\big)=D(a)\cdot D(b)+a\cdot D^2(b)=D(a)\cdot D(b)\quad\forall a,b\in\mathcal{A}.$$

(c) A projector D, that is, D is an algebra morphism and $D^2=D$. Then

$$D\big(a\cdot D(b)\big)=D(a)\cdot D^2(b)=D(a)\cdot D(b)\quad\forall a,b\in\mathcal{A}.$$

Now, we can give a simple example of a left Loday quasi-algebroid which does not admit a right Loday quasi-algebroid structure.

Example 15. Consider $\mathcal{F}=\Omega(\mathbb{R}^6)$, which is an $\mathbb{R}$-algebra with the exterior product $\wedge$ and, moreover, a $\mathcal{C}^\infty(\mathbb{R}^6)$-module (i.e. $\mathcal{R}=\mathbb{R}$, $\mathcal{A}=\mathcal{C}^\infty(M)$). Define

$$[\![\alpha,\beta]\!]=d(\alpha)\wedge\beta-\beta\wedge d(\alpha)=\big(1-(-1)^{|\beta\|\alpha+1|}\big)d(\alpha)\wedge\beta.$$

It is immediate that $(\Omega(\mathbb{R}^6),[\![\ ,\ ]\!])$ is a left Loday algebra, as d is a square-zero operator (see (b) above). Now, we have, for any $\alpha,\beta\in\Omega(\mathbb{R}^6)$, $f\in\mathcal{C}^\infty(\mathbb{R}^6)$,

$$[\![\alpha,f\cdot\beta]\!]=\big(1-(-1)^{|\beta\|\alpha+1|}\big)d(\alpha)\wedge f\cdot\beta=f\cdot\big(1-(-1)^{|\beta\|\alpha+1|}\big)d(\alpha)\wedge\beta=f\cdot[\![\alpha,\beta]\!].$$

Thus, on the other hand, if $f=x_6$, $\alpha=dx_1\wedge dx_2\wedge dx_3\wedge dx_4$, $\beta=dx_5$,

$$[\![f\cdot\alpha,\beta]\!]=\big(1-(-1)^{|1\|5|}\big)d\big(x_6\cdot dx_1\wedge dx_2\wedge dx_3\wedge dx_4\big)\wedge dx_5=-2dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dx_5\wedge dx_6$$

but

$$f \cdot [\![\alpha, \beta]\!] = \big(1 - (-1)^{|1\|5|}\big) x_6 \cdot d\big(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\big) \wedge dx_5 = 0$$

and there is no $q^L_{\Omega(\mathbb{R}^6)} : \Omega(\mathbb{R}^6) \to \mathfrak{X}(\mathbb{R}^6)$ such that

$$q^L_{\Omega(\mathbb{R}^6)}(\alpha)(f) \cdot dx_5 = -2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6.$$

Thus, $(\Omega(\mathbb{R}^6), [\![\ ,\ ]\!])$ has a left Loday quasi-algebroid structure, with anchor $q^L_{\Omega(\mathbb{R}^6)} \equiv 0$, but it does not admit a right Loday quasi-algebroid structure. Note that $q^L_{\Omega(\mathbb{R}^6)}$ is (trivially) tensorial.

The following, less trivial, example was suggested to us by Y. Sheng. It shows that the kind of structures we are considering can appear in the more general context of higher order Courant algebroids (although here we just take $n = 1$ for simplicity) through the associated Dorfman bracket; see [11].

Example 16. Let M be a differential manifold. Consider the vector bundle $TM \oplus T^*M$ whose sections are endowed with the Dorfman bracket:

$$[\![\ ,\ ]\!] : TM \oplus T^*M \times TM \oplus T^*M \longrightarrow TM \oplus T^*M, \quad (X + \alpha, Y + \beta) \longmapsto [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha.$$

Then we have a left Loday algebra, as $[\![\ ,\ ]\!]$ is clearly $\mathbb{R}$-bilinear and

$$\begin{aligned}
&[\![[\![X + \alpha, Y + \beta]\!], Z + \gamma]\!] + [\![Y + \beta, [\![X + \alpha, Z + \gamma]\!]]\!] \\
&\quad = [[X, Y], Z] + \mathcal{L}_{[X,Y]}\gamma - \iota_Z d\big(\mathcal{L}_X \beta - \iota_Y d\alpha\big) + [Y, [X, Z]] + \mathcal{L}_Y\big(\mathcal{L}_X \gamma - \iota_Z d\alpha\big) - \iota_{[X,Z]} d\beta \\
&\quad = [X, [Y, Z]] + \mathcal{L}_{[X,Y]}\gamma + \mathcal{L}_Y\big(\mathcal{L}_X \gamma\big) - \iota_Z d\big(\mathcal{L}_X \beta\big) - \iota_{[X,Z]} d\beta + \iota_Z d\big(\iota_Y d\alpha\big) - \mathcal{L}_Y\big(\iota_Z d\alpha\big) \\
&\quad = [X, [Y, Z]] + \mathcal{L}_X\big(\mathcal{L}_Y \gamma\big) - \mathcal{L}_X \iota_Z d\beta - \iota_{[Y,Z]} d\alpha \\
&\quad = [X, [Y, Z]] + \mathcal{L}_X\big((\mathcal{L}_Y \gamma) - \iota_Z d\beta\big) - \iota_{[Y,Z]} d\alpha \\
&\quad = [\![X + \alpha, [\![Y + \beta, Z + \gamma]\!]]\!],
\end{aligned}$$

for all $X + \alpha$, $Y + \beta$, $Z + \gamma \in TM \oplus T^*M$. Moreover, $\mathrm{ad}^L_{X+\alpha} \in \mathrm{QDer}_{\mathbb{R}}(TM \oplus T^*M)$. Let $f \in \mathcal{C}^\infty(M)$ and $Y + \beta \in TM \oplus T^*M$. Then,

$$\begin{aligned}
\big[\mathrm{ad}^L_{X+\alpha}, \mu_f\big](Y + \beta) &= [\![X + \alpha, f \cdot (Y + \beta)]\!] - f \cdot [\![X + \alpha, Y + \beta]\!] \\
&= [X, f \cdot Y] + \mathcal{L}_X(f\alpha) - \iota_{f \cdot Y} d\alpha - f[X, Y] - f\mathcal{L}_X \beta + f\iota_y d\alpha \\
&= X(f) \cdot Y + \mathcal{L}_X(f\beta) - f\mathcal{L}_X \beta + f\iota_Y d\alpha - \iota_{fY} d\alpha \\
&= X(f) \cdot Y + X(f) \cdot \beta = X(f) \cdot (Y + \beta) = \mu_{X(f)}(Y + \beta) \\
&= \widehat{\mathrm{ad}^L_{X+\alpha}}(f) \cdot (Y + \beta);
\end{aligned}$$

so $(TM \oplus T^*M, [\![\ ,\ ]\!])$ is a left Loday quasi-algebroid, with anchor map the projection onto the first factor:

$$q^L_{TM \oplus T^*M}(X + \alpha) = \widehat{\mathrm{ad}^L_{X+\alpha}} = X.$$

Note that in this case the anchor is tensorial: if $f, g \in \mathcal{C}^\infty(M)$ and $X + \alpha \in TM \oplus T^*M$, then

$$\big[q^L_{TM \oplus T^*M}, \mu_f\big](X + \alpha)(g) = \widehat{\mathrm{ad}^L_{f \cdot (X+\alpha)}}(g) - f \cdot \widehat{\mathrm{ad}^L_{X+\alpha}}(g) = 0,$$

so $(TM \oplus T^*M, [\![\ ,\ ]\!])$ is indeed a left Loday algebroid. However, for $\mathrm{ad}^R_{X+\alpha}$ we find

$$\begin{aligned}
\big[\mathrm{ad}^R_{X+\alpha}, \mu_f\big](Y + \beta) &= B\big(f(Y + \beta), X + \alpha\big) - f \cdot B(Y + \beta, X + \alpha) \\
&= [f \cdot Y, X] + \mathcal{L}_{f \cdot Y}\alpha - \iota_X d(f \cdot \beta) - f \cdot [X, Y] - f\mathcal{L}_Y \alpha + f\iota_X d\beta \\
&= -X(f) \cdot Y + \mathcal{L}_{f \cdot Y}\alpha - \iota_X d(f \cdot \beta) - f\mathcal{L}_Y \alpha + f\iota_X d\beta
\end{aligned}$$

and the term $\mathcal{L}_{f \cdot Y}\alpha - \iota_X d(f \cdot \beta) - f\mathcal{L}_Y \alpha + f\iota_X d\beta$ clearly spoils the possibility that $\mathrm{ad}^R_{X+\alpha}$ is a quasi-derivation.

5 Left omni-Loday algebroids and omni-Lie algebroids

Having established the non-triviality of left Loday quasi-algebroids, we now turn to the question of whether an analogue of Weinstein's omni-Lie algebra exists for these structures. As before, let $\mathcal{A}$ be an associative algebra, commutative and with unit element $1_{\mathcal{A}}$ over a ring $\mathcal{R}$ that is commutative and with unit element $1_{\mathcal{R}}$. Also, let $\mathcal{F}$ be a faithful $\mathcal{A}$-module.

Definition 17. Consider the product space $\mathfrak{gl}(\mathcal{F}) \times \mathcal{F}$ and define the bracket

$$\{\ ,\ \} : \big(\mathfrak{gl}(\mathcal{F}) \times \mathcal{F}\big) \times \big(\mathfrak{gl}(\mathcal{F}) \times \mathcal{F}\big) \longrightarrow \mathfrak{gl}(\mathcal{F}) \times \mathcal{F}, \quad \big((\Phi, X), (\Psi, Y)\big) \longmapsto \big\{(\Phi, X), (\Psi, Y)\big\} := ([\Phi, \Psi], \Phi Y),$$

where $[\ ,\ ]$ is the commutator of endomorphisms.

Remark 18. It is straightforward to check that $\{\ ,\ \}$ is $\mathcal{R}$-bilinear. However, it does not satisfy Jacobi's identity (here $\circlearrowleft$ denotes cyclic sum), as we have

$$\circlearrowleft \big\{(\Phi, X), \big\{(\Psi, Y), (\Upsilon, Z)\big\}\big\} = \big(0, \Phi(\Psi Z) + \Upsilon(\Phi Y) + \Psi(\Upsilon X)\big).$$

As stated in the introduction, the bracket $\{\ ,\ \}$ satisfies instead of the left Leibniz identity the following:

$$\big\{(\Phi, X), \big\{(\Psi, Y), (\Upsilon, Z)\big\}\big\} = \big\{\big\{(\Phi, X), (\Psi, Y)\big\}, (\Upsilon, Z)\big\} + \big\{(\Psi, Y), \big\{(\Phi, X), (\Upsilon, Z)\big\}\big\}.$$

Now, let $B : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be an $\mathcal{R}$-bilinear form. Define the "graph" of B as

$$\mathcal{F}_B := \big\{\big(\mathrm{ad}^L_X, X\big) : X \in \mathcal{F}\big\} \subseteq \mathfrak{gl}(\mathcal{F}) \times \mathcal{F},$$

where

$$\mathrm{ad}^L_X : \mathcal{F} \longrightarrow \mathcal{F}, \quad Y \longmapsto B(X, Y).$$

Proposition 19. *The graph $\mathcal{F}_B$ is closed under $\{\ ,\ \}$ if and only if $(\mathcal{F}, B)$ is a left Loday algebra. Moreover, if B is antisymmetric, $\mathcal{F}_B$ is closed if and only if $(\mathcal{F}, B)$ is a Lie algebra.*

Proof. $\mathcal{F}_B$ is closed with respect to $\{\ ,\ \}$ if, and only if, for all $X, Y \in \mathcal{F}$ we have

$$\big\{\big(\mathrm{ad}^L_X, X\big), \big(\mathrm{ad}^L_Y, Y\big)\big\} = \big(\big[\mathrm{ad}^L_X, \mathrm{ad}^L_Y\big], \mathrm{ad}^L_X(Y)\big) = \big(\mathrm{ad}^L_{B(X,Y)}, B(X, Y)\big),$$

that is, for all $Z \in \mathcal{F}$,

$$\big[\mathrm{ad}^L_X, \mathrm{ad}^L_Y\big](Z) = \mathrm{ad}^L_{B(X,Y)}(Z),$$

or

$$B\big(X, B(Y, Z)\big) - B\big(Y, B(X, Z)\big) = B\big(B(X, Y), Z\big) \quad \forall X, Y, Z \in \mathcal{F},$$

or equivalently

$$B\big(B(X, Y), Z\big) + B\big(Y, B(X, Z)\big) = B\big(X, B(Y, Z)\big) \quad \forall X, Y, Z \in \mathcal{F},$$

which is the left Leibniz identity, that is, $(\mathcal{F}, B)$ is a left Loday algebra. $\square$

For left Loday quasi-algebroids, we have the following.

Theorem 20. *Let $B : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be an $\mathcal{R}$-bilinear form and $\rho : \mathcal{F} \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A})$ a morphism of $\mathcal{R}$-modules. Then, $(\mathcal{F}, B)$ is a left Loday quasi-algebroid with anchor map ρ, if and only if $\mathcal{F}_B$ is closed with respect to $\{\ ,\ \}$ and B is such that*

$$B(X, f \cdot Z) = f \cdot B(X, Z) + \rho(X)(f) \cdot Z,$$

(i.e. $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$).

Proof. If $(\mathcal{F}, B)$ is a left Loday quasi-algebroid, it is also a left Loday algebra and then, by Proposition 19, $\mathcal{F}_B$ is closed under $\{\ ,\ \}$. On the other hand, the condition of being quasi-algebroid implies that for all $X, Z \in \mathcal{F}$ and for all $f \in \mathcal{A}$, we have

$$\big[\mathrm{ad}^L_X, \mu_f\big](Z) = \rho(X)(f) \cdot Z,$$

that is,

$$B(X, f \cdot Z) = f \cdot B(X, Z) + \rho(X)(f) \cdot Z.$$

For the second implication, consider $\mathcal{F}_B$ closed with respect to $\{\ ,\ \}$, so $(\mathcal{F}, B)$ is a left Loday algebra (see Proposition 19). Moreover, for all $X, Z \in \mathcal{F}$ and for all $f \in \mathcal{A}$,

$$\begin{aligned}\big[\mathrm{ad}^L_X, \mu_f\big](Z) &= B(X, f \cdot Z) - f \cdot B(X, Z) = f \cdot B(X, Z) + \rho(X)(f) \cdot Z - f \cdot B(X, Z) \\ &= \rho(X)(f) \cdot Z = \mu_{\rho(X)(f)}(Z),\end{aligned}$$

so

$$\big[\mathrm{ad}^L_X, \mu_f\big] = \mu_{\rho(X)(f)}$$

that is, ad^L_X is a quasi-derivation for all $X \in \mathcal{F}$. Thus, $(\mathcal{F}, B)$ is a left quasi-algebroid with anchor map ρ. □

Let us try to get rid of the "quasi" prefix.

Theorem 21. *Let $\mathcal{F}$ be a free module of rank $k > 1$, $B : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ an $\mathcal{R}$-bilinear form and $\rho : \mathcal{F} \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A})$ a morphism of $\mathcal{R}$-modules. Suppose that $\mathcal{F}_B$ is closed with respect to $\{\ ,\ \}$ and that B is such that*

(a) $B(X, g \cdot Z) = g \cdot B(X, Z) + \rho(X)(g) \cdot Z,$
(b) $B(f \cdot X, Z) = f \cdot B(X, Z) - \rho(Z)(f) \cdot X$

(i.e. $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$). Then $(\mathcal{F}, B, \rho)$ is a left Loday algebroid.

Proof. On one hand, applying (a) then (b),

$$B(f \cdot X, g \cdot Z) = g \cdot B(f \cdot X, Z) + \rho(f \cdot X)(g) \cdot Z = gf \cdot B(X, Z) - g\rho(Z)(f) \cdot X + \rho(f \cdot X)(g) \cdot Z,$$

and on the other hand, first applying (b),

$$B(f \cdot X, g \cdot Z) = f \cdot B(X, g \cdot Z) - \rho(g \cdot Z)(f) \cdot X = fg \cdot B(X, Z) + f\rho(X)(g) \cdot Z - \rho(g \cdot Z)(f) \cdot X.$$

Therefore, $-g\rho(Z)(f) \cdot X + \rho(f \cdot X)(g) \cdot Z = f\rho(X)(g) \cdot Z - \rho(g \cdot Z)(f) \cdot X$, or equivalently

$$\big(\rho(g \cdot Z) - g\rho(Z)\big)(f) \cdot X + \big(\rho(f \cdot X) - f\rho(X)\big)(g) \cdot Z = 0. \tag{5.1}$$

Let $\{Y_j\}_{j \in I}$ be a basis of $\mathcal{F}$. Taking $X = Y_i$ and $Z = Y_j$ for some distinct $i, j \in I$ in (5.1), we have

$$\big(\rho(g \cdot Y_j) - g\rho(Y_j)\big)(f) \cdot Y_i + \big(\rho(f \cdot Y_i) - f\rho(Y_i)\big)(g) \cdot Y_j = 0,$$

so

$$\big(\rho(g \cdot Y_j) - g\rho(Y_j)\big)(f) = 0 = \big(\rho(f \cdot Y_i) - f\rho(Y_i)\big)(g) \tag{5.2}$$

for all $f, g \in \mathcal{A}$.

Now, let $X = \sum_{j \in I} k_j \cdot Y_j$ and $f \in \mathcal{A}$. By the $\mathcal{R}$-linearity of ρ and (5.2),

$$\begin{aligned}\rho(f \cdot X) &= \rho\Bigg(f \sum_{j \in I} k_j \cdot Y_j\Bigg) = \rho\Bigg(\sum_{j \in I} fk_j \cdot Y_j\Bigg) = \sum_{j \in I} \rho\big(fk_j \cdot Y_j\big) = \sum_{j \in I} fk_j \cdot \rho\big(Y_j\big) \\ &= f \sum_{j \in I} k_j \cdot \rho\big(Y_j\big) = f \sum_{j \in I} \rho\big(k_j \cdot Y_j\big) = f\rho\Bigg(\sum_{j \in I} k_j \cdot Y_j\Bigg) = f \cdot \rho(X),\end{aligned}$$

that is, ρ is tensorial. Thus, $(\mathcal{F}, B, \rho)$ is a left Loday algebroid. □

Remark 22. However, we can not say anything about the converse, as Example 15 shows (there, we have a left Loday algebroid and the first condition (a) above is trivially satisfied while (b) is not).

We also can avoid the "quasi" prefix if we add the condition of antisymmetry to B, thus entering into the realm of Lie structures.

Theorem 23. *Let $\mathcal{F}$ be a free module of rank $k > 1$, $B : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ an $\mathcal{R}$-bilinear form, and $\rho : \mathcal{F} \to \mathrm{Der}_{\mathcal{R}}(\mathcal{A})$ a morphism of $\mathcal{R}$-modules. Then, $(\mathcal{F}, B, \rho)$ is a Lie algebroid if and only if $\mathcal{F}_B$ is closed with respect to $\{\ ,\ \}$, B is antisymmetric and, for all $X, Z \in \mathcal{F}$ and $f \in \mathcal{A}$, the following holds:*

$$B(X, f \cdot Z) = f \cdot B(X, Z) + \rho(X)(f) \cdot Z$$

(that is, $\mathrm{ad}_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{A})$).

Proof. If $(\mathcal{F}, B, \rho)$ is a Lie algebroid, $(\mathcal{F}, B)$ is a Lie algebra, that is, $(\mathcal{F}, B)$ is a left (and right) Loday algebra and B is antisymmetric, so Proposition 19 tells us that $\mathcal{F}_B$ is closed with respect to $\{\ ,\ \}$. Now, let $X, Z \in \mathcal{F}$, $f \in \mathcal{A}$; then we have

$$\left[\mathrm{ad}^L_X, \mu_f\right](Z) = \mu_{\rho(X)(f)}(Z),$$

which is the same as

$$B(X, f \cdot Z) = f \cdot B(X, Z) + \rho(X)(f) \cdot Z.$$

If now is $\mathcal{F}_B$ closed with respect to $\{\ ,\ \}$, Proposition 19 again tells us that $(\mathcal{F}, B)$ is a left Loday algebra, but as B is also antisymmetric, $(\mathcal{F}, B)$ is a Lie algebra.

On the other hand, the hypothesis of Theorem 20 is satisfied, so we know that ad_X is a quasi-derivation for all $X \in \mathcal{F}$ and $(\mathcal{F}, B)$ is a Lie quasi-algebroid with anchor map ρ.

To finish, let us check (see Theorem 21) that for all $X, Y, Z \in \mathcal{F}$ and for all $f \in \mathcal{A}$ the following holds:

$$B(f \cdot X, Z) = f \cdot B(X, Z) - \rho(Z)(f) \cdot X.$$

But, by the antisymmetry of B,

$$B(f \cdot X, Z) = -B(Z, f \cdot X) = -f \cdot B(Z, X) - \rho(Z)(f) \cdot X = f \cdot B(X, Z) - \rho(Z)(f) \cdot X.$$

So $(\mathcal{F}, B, \rho)$ is a Lie algebroid. $\square$

The preceding results motivate the following definition.

Definition 24. Let $\mathcal{A}$ be an associative, commutative algebra with unit element $1_{\mathcal{A}}$ over a commutative ring with unit element $1_{\mathcal{R}}$. Let $\mathcal{F}$ be a free $\mathcal{A}$-module of rank $k > 1$. We call $(\mathfrak{gl}(\mathcal{F}) \times \mathcal{F}, \{\ ,\ \})$ the left omni-Loday algebroid determined by $\mathcal{F}$.

Remark 25. Note that if $(\mathcal{F}, B, \rho)$ is a left Loday algebroid then, in particular, it is a left Loday quasi-algebroid and thus $\mathcal{F}_B \subset \mathfrak{gl}(\mathcal{F}) \times \mathcal{F}$ is closed with respect to $\{\ ,\ \}$, as by Theorem 20: every left Loday algebroid can be seen as a closed subspace of left omni-Loday algebroid.

Remark 26. In the case of Lie algebroids, we have the same situation as in the preceding remark: given an $\mathcal{R}$-bilinear $\mathcal{F}$-valued form $B : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ such that it is antisymmetric and satisfies $\mathrm{ad}^L_X \in \mathrm{QDer}_{\mathcal{R}}(\mathcal{F})$, by Theorem 23 there is a correspondence between Lie algebroids $(\mathcal{F}, B, \rho)$ and closed subspaces $\mathcal{F}_B$, but this time given by an "if and only if" statement. Thus, we could call $(\mathfrak{gl}(\mathcal{F}) \times \mathcal{F}, \{\ ,\ \})$ an omni-Lie algebroid as well.

It is worth noting that a different definition for omni-Lie algebroids (based on the notion of Courant structures on the direct sum of the gauge Lie algebroid and the bundle of jets of a vector bundle E over a manifold M) has been presented very recently in [1]. It would be interesting to know if this definition is equivalent to ours.

Acknowledgments The authors want to express their gratitude to Prof. M. K. Kinyon and Prof. Y. Sheng for their useful comments and for providing references to previous work on Loday algebras. The second author's work was supported by a CONACyT grant, project CB-2007-1 code J2-78791.

References

[1] Z. Chen and Z.-J. Liu, *Omni-Lie algebroids*, J. Geom. Phys., 60 (2010), pp. 799–808.
[2] J. Grabowski, *Quasi-derivations and QD-algebroids*, Rep. Math. Phys., 52 (2003), pp. 445–451.
[3] M. K. Kinyon, *Leibniz algebras, Lie racks, and digroups*, J. Lie Theory, 17 (2007), pp. 99–114.
[4] M. K. Kinyon and A. Weinstein, *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*, Amer. J. Math., 123 (2001), pp. 525–550.
[5] Y. Kosmann-Schwarzbach, *From Poisson algebras to Gerstenhaber algebras*, Ann. Inst. Fourier (Grenoble), 46 (1996), pp. 1243–1274.
[6] ———, *Loday algebra (Leibniz algebra)*, in Encyclopaedia of Mathematics, Supplement II, Kluwer, Dordrecht, 2000, pp. 318–319.
[7] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. H. Poincaré Phys. Théor., 53 (1990), pp. 35–81.
[8] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. (2), 39 (1993), pp. 269–293.
[9] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, vol. 213 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2005.
[10] F. Ongay, *An example of a coquecigrue embedded in* $\mathbb{R}^4$, Bull. Allahabad Math. Soc., 23 (2008), pp. 301–309.
[11] Y. Sheng, *On higher-order courant brackets*. Available at http://arxiv.org/abs/1003.1350, 2010.
[12] Y. Sheng and C. Zhu, *Semidirect products of representations up to homotopy*. To appear in the Pacific J. Math.
[13] ———, *Integration of semidirect product lie 2-algebras*. Available at http://arxiv.org/abs/1003.1348, 2010.
[14] M. Stiénon and P. Xu, *Modular classes of Loday algebroids*, C. R. Math. Acad. Sci. Paris, 346 (2008), pp. 193–198.
[15] A. Weinstein, *Omni-Lie algebras*, Sūrikaisekikenkyūsho Kōkyūroku, (2000), pp. 95–102. Microlocal analysis of the Schrödinger equation and related topics (Japanese) (Kyoto, 1999).