On Calabi-Bernstein results for maximal surfaces in Lorentzian products

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- The function Θ, $\Theta : \Sigma \to \mathbb{R}$, defined by

$$\Theta = \langle T, N \rangle.$$

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A global theorem for complete maximal surfaces in $\mathbb{M} \times_{-} \mathbb{R}$

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Theorem

Let $f : \Sigma \to \mathrm{M} \times\_ \mathbb{R}$ be a complete maximal surface with $K_{\mathrm{M}} \geq 0$ along $\pi_{\mathrm{M}}(f(\Sigma))$. Then,

i) Σ is a totally geodesic surface.

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- A Riemannian manifold is called parabolic if any positive superharmonic function is constant. Or, equivalently, if any negative subharmonic function on the surface is constant.
- **Parabolicity Criterium (Ahlfors [1] and Blanc-Fiala-Huber [8])**
  Any complete Riemannian surface with non-negative Gauss curvature is parabolic.

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By the chain rule,

$$\begin{aligned} \Delta\Phi(\Theta) = &\quad \Phi'(\Theta)\,\Delta\Theta + \Phi''(\Theta)\,\|\nabla\Theta\|^2 \\ = &\quad \frac{-\Theta^2(\Theta^2-1)K_{\mathrm{M}}(\pi) - \|A\|^2}{\Theta^3} \geq 0 \end{aligned}$$

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Corollary

Let $f : \Sigma \to \mathrm{M} \times_{-} \mathbb{R}$ be a complete maximal surface with $K_{\mathrm{M}} \geq 0$ along $\pi_{\mathrm{M}}(f(\Sigma))$. Then, Σ endowed with the metric induced by the immersion f is a parabolic surface.

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  Moreover, as any totally geodesic surface in $\mathbb{L}^3$ must be a plane, the Calabi-Bernstein theorem is a consequence of the theorem.

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  Example: There exist complete maximal, but non totally geodesic graphs when M is the hyperbolic plane $\mathbb{H}^2$.

  To see this, we need a duality result.

A duality result

Theorem

Let M be an orientable surface and let $\Omega \subseteq \mathrm{M}$ be a simply connected domain. There exists a $\mathcal{C}^2$ solution with non constant gradient of the minimal surface equation on Ω

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

if and only if there exists a $\mathcal{C}^2$ solution with non constant gradient of the maximal surface equation on Ω

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- Alías and Palmer, [5], proved this result in the case $\mathrm{M} = \mathbb{R}^2$.

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$$D\omega = J\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

Dual graphs

- Let $f_u(x) = (x, u(x))$, (M, g_u), be a minimal graph over $\mathrm{M} \times_+ \mathbb{R}$ and $f_\omega(x) = (x, \omega(x))$, (M, g_ω), its maximal dual graph over $\mathrm{M} \times_- \mathbb{R}$, where M is a simply connected, orientable, complete surface.

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Comparing the metric.

- Let $G \subset \mathrm{M}$ be the set $G = \{x \in \mathrm{M} : Du(x) \neq 0\}$. Then $\{E_1, E_2\}$, where $E_1 = \frac{Du}{|Du|}$ $E_2 = \frac{D\omega}{|D\omega|}$, is an orthonormal basis of $\mathcal{X}(G)$.

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$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_u = \begin{pmatrix} 1 + |Du|^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_\omega = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+|Du|^2} \end{pmatrix}$$

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$$\boxed{g_\omega \leq g \leq g_u}$$

- (M, g_u) is always a complete minimal graph.
- If $|Du|^2$ is bounded, (M, g_ω) is a complete maximal graph.

Dual graphs

Comparing the second fundamental form.

- Let $X \in \mathcal{X}(M)$,

$$A_u X = -\frac{1}{\sqrt{1+|Du|^2}} D_X Du + \frac{g(D_X Du, Du)}{(1+|Du|^2)^{3/2}} Du,$$

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$$\boxed{\begin{aligned} A_\omega X &= -J(D_X Du), \\ A_u X &= J(D_X D\omega) \end{aligned}}$$

- A maximal graph is non totally geodesic if and only if Du is not constant.

The assumptions on K_{M}

- The assumption $K_{\mathrm{M}} \geq 0$ is necessary.

  Example: There exist complete maximal, but non totally geodesic graphs when M is the hyperbolic plane $\mathbb{H}^2$.

  Consider the minimal graph over $\mathbb{H}^2 \times_+ \mathbb{R}$ determined by the function

$$u(x, y) = \log(x^2 + y^2).$$

  (Montaldo and Onnis [10])

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- $|Du|^2 = \frac{4y^2}{x^2+y^2} \leq 4$ $\rightsquigarrow$ The corresponding maximal dual graph is complete.

A local result on maximal surfaces in $\mathrm{M} \times_{-} \mathbb{R}$

Theorem

Let M be an analytic Riemannian surface and let $f : \Sigma \to \mathrm{M} \times_{-} \mathbb{R}$ be a maximal surface such that $K_{\mathrm{M}}(\pi) \geq 0$. Let $p \in \Sigma$, and $R > 0$ be such that the geodesic disc of radius R about p satisfies $D(p, R) \subset\subset \Sigma$. Then for all $0 < r < R$,

$$0 \leq \int_{D(p,r)} \|A_q\|^2 dA \leq c_r \frac{L(r)}{r \log(R/r)}$$

where $L(r)$ denotes the length of $\partial D(p, r)$, and

$$c_r = \frac{\pi^2}{4} \frac{(1 + a_r^2)^2}{a_r \mathrm{arctan}(a_r)} > 0.$$

Here, a_r is a positive number such that $-a_r \leq \Theta(q) \leq -1$ is verifies for all $q \in D(p, r)$.

Proof of the theorem

i) $K_\Sigma \geq 0$ as in the global theorem.

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Lemma (Alías, Palmer [4])

Let Σ be an analytic Riemannian surface with $K_\Sigma \geq 0$. Let $u \in \mathcal{C}^\infty(\Sigma)$ which satisfies

$$u\Delta u \geq 0$$

on Σ. Then, for $0 < r < R$,

$$\int_{D(p,r)} u\Delta u \leq \frac{2L(r)}{r\log(R/r)} \sup_{D(p,R)} u^2,$$

where p is a fixed point in Σ and $D_r \subset D_R \subset\subset \Sigma$.

Proof of the theorem

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Corollary

Let M be an analytic Riemannian surface. Then, the only complete maximal surfaces, $f : \Sigma \to \mathrm{M} \times_{-} \mathbb{R}$, with $K_{\mathrm{M}} \geq 0$ are the totally geodesic ones.

Entire maximal graphs

Proposition (Alías, Romero, Sánchez [7])

Consider $\mathrm{M} \times_{-} \mathbb{R}$ where M is simply connected. Then every complete spacelike surface Σ is an entire graph. Moreover, M is compact if and only if Σ is compact too.

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- The Calabi-Bernstein theorem can be formulated in two equivalent ways:
  - The only complete maximal surfaces in $\mathbb{L}^3$ are the affine spacelike graphs.
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Entire maximal graphs

Theorem

Let M be a complete surface with $K_{\mathrm{M}} \geq 0$ and let $f_u : \mathrm{M} \to \mathrm{M} \times\_ \mathbb{R}$ be an entire maximal graph. Then,

i) The graph is totally geodesic.

ii) If, in addition, M is not a flat surface, the graph is a slice.

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- Superharmonic functions are invariant under conformal changes of metric in the 2-dimensional case.

References

[1] Ahlfors, L. V., *Sur le type d'une surface de Riemann*, C. R. Acad. Sc. Paris **201** (1935), 30–32.

[2] Albujer, A. L. and Alías, L. J., *On Calabi-Bernstein results for maximal surfaces in Lorentzian products*, preprint 2005.

[3] Alías, L. J. and Dajczer, M., *Bernstein-type results in* $\mathbb{M}^2 \times \mathbb{R}$, preprint 2005.

[4] Alías, L. J. and Palmer, B., *Zero mean curvature surfaces with non-negative curvature in flat Lorentzian 4-spaces*, R. Soc. Lond. Proc. Ser. A. Math. Phys. Eng. Sci. **455** (1999), 631–636.

[5] Alías, L. J. and Palmer, B., *A duality result between the minimal surface equation and the maximal surface equation*, An. Acad. Brasil. Ciênc. **73** (2001), 161–164.

[6] Alías, L.J. and Palmer, B., *On the Gaussian curvature of maximal surfaces and the Calabi-Bernstein theorem*, Bull. London Math. Soc. **33** (2001), 454–458.

[7] Alías, L. J., Romero, A. and Sánchez, M., *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **27** (1995), 71–84.

[8] Huber, A., *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

[9] Latorre, J. M. and Romero, A., *New examples of Calabi-Bernstein problems for some nonlinear equations*, Differential Geom. Appl. **15** (2001), 153–163.

[10] Montaldo, S. and Onnis, I. I., *Invariant surfaces in* $\mathbb{H}^2 \times \mathbb{R}$ *with constant (mean or Gauss) curvature*, Proceedings of XIII Workshop on Geometry and Physics, Murcia 2004, Publ. de la RSME **9** (2005), 91–103.

[11] Romero, A., *Simple proof of Calabi-Bernstein's theorem on maximal surfaces*, Proc. Amer. Math. Soc. **124** (1996), 1315–1317.

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