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Walker metrics: Applications to Osserman manifolds

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WALKER METRICS

Definition

(M^n, g) is said to be a *Walker manifold* if it admits a degenerate parallel plane field.

Examples:

- Plane wave metrics
- Weakly-irreducible pseudo-Riemannian metrics
- Two-step nilpotent L. g. with degenerate center
- Metrics on tangent and cotangent bundles
  $\longrightarrow$ Complete lifts to TM
  $\longrightarrow$ Riemann extensions to T^*M
- Para-Kähler and Hypersymplectic structures.

Underlying structure of some geometric problems:

- Nonuniqueness of the metric for the Levi Civita connection.
- Einstein hypersurfaces in indefinite space forms.
- Curvature conditions:
  $\longrightarrow$ Self-dual K. and A.K. Einstein metrics,
  $\longrightarrow$ Harmonic manifolds, **Osserman metrics**, ...

CANONICAL FORM OF A WALKER METRIC

(M^n, g) admits a degenerate parallel plane field $\mathcal{D}^r$ $\longleftrightarrow$ there are coordinates $(x_1, \ldots, x_n)$ such that

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & I_r \\ 0 & A & H \\ I_r & {}^tH & B_{r\times r} \end{pmatrix},$$

A, B symmetric

A, H independent of $(x_1, \ldots, x_r)$.

✓ $\mathcal{D}$ is spanned by $\{\partial_1, \ldots, \partial_r\}$.

- Special cases:

$n = 4$, $r = 2$

$$g_{(x_1,\ldots,x_4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix},$$

a, b, c functions of $(x_1, \ldots, x_4)$.

There exists two orthogonal parallel null vectors

$$a \equiv 0, \quad c \equiv 0, \quad b \equiv b(x_3, x_4)$$

✓ Any such metric globally defined on $\mathbb{R}^4$ is geodesically complete.

OSSERMAN METRICS

(M^n, g): pseudo-Riemannian manifold with curvature tensor R. For a point $p \in M$ the *Jacobi operator* is defined by

$$R_X = R(X, \cdot\,)X, \qquad X \in T_pM.$$

Definitions

- (M, g) is *globally Osserman* if the eigenvalues of the Jacobi operators are constant on $S^{\pm}(TM)$.

- (M, g) is *pointwise Osserman* if the eigenvalues of R_X are independent of the direction X, but may change from point to point.

✓ Two-point homogeneous→globally Osserman

- (M, g) is *(pointwise) Jordan-Osserman* if the Jordan normal form of the Jacobi operators is (pointwise) constant on $S^{\pm}(TM)$.

→ The eigenvalue structure does not determine the Jordan normal form in the indefinite setting.

OSSERMAN METRICS. AN OVERVIEW

Riemannian case

- Riemannian Osserman spaces are locally two-point homogeneous (dim $M \neq 16$).
- Osserman algebraic curvature tensors exist which do not correspond to any symmetric space.

Lorentzian case

- Osserman algebraic curvature tensors are of constant curvature.

Higher signature

- Nonsymmetric and non locally homogeneous Osserman metrics exist.
- Partial classification is available under additional conditions:
  - $\longrightarrow$ $dim M = 4$ + diagonalizable Jacobi operators
  - $\longrightarrow$ $dim M = 4 + \nabla R = 0$
  - $\longrightarrow$ Diagonalizable Jacobi operators with exactly two-distinct eigenvalues

SOME ALGEBRAIC FACTS

- Spacelike and timelike Osserman conditions are equivalent at the algebraic level.

- Spacelike and timelike Jordan-Osserman conditions are not equivalent.

- The spacelike Jacobi operators of a spacelike Jordan-Osserman algebraic curvature tensor are necessarily diagonalizable whenever $p < q$ and the signature is $(-\overset{p}{\cdots}-,+\overset{q}{\cdots}+)$.

- Let $\mathcal{A}$ be a self-adjoint map of $\mathbb{R}^{(l,l)}$. There exist Jordan-Osserman algebraic curvature tensors in $\mathbb{R}^{(p,p)}$, with $p = 2^{l+1}$ such that

$$R_X = \left(\begin{array}{c|c} \langle X, X \rangle \mathcal{A} & 0 \\ \hline 0 & 0 \end{array} \right).$$

FOUR-DIMENSIONAL OSSERMAN METRICS

Osserman $\longrightarrow$ Einstein

$$\left\{\begin{array}{c}\text{Osserman}\\ +\\ dim M = 3\end{array}\right\} \longrightarrow \text{constant sectional curvature}$$

Special features of dimension four

- First nontrivial case
- $\left\{\begin{array}{c}\text{pointwise}\\ \text{Osserman}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Einstein}\\ \text{self-dual (or anti-self-dual)}\end{array}\right.$
- All possible Jordan normal forms are realized at the algebraic level

$$X^\perp \xrightarrow{R_X} X^\perp$$

$$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{pmatrix} \quad \begin{pmatrix}\alpha & \beta & 0\\ -\beta & \alpha & 0\\ 0 & 0 & \gamma\end{pmatrix} \quad \begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 1 & \beta\end{pmatrix} \quad \begin{pmatrix}\alpha & 0 & 0\\ 1 & \alpha & 0\\ 0 & 1 & \alpha\end{pmatrix}$$

Type Ia: Type Ib: Type II: Type III:

FOUR-DIMENSIONAL OSSERMAN METRICS

Type Ia Osserman metrics are classified:

$\longrightarrow$ real space forms

$\longrightarrow$ complex space forms

$\longrightarrow$ paracomplex space forms

Type Ib Osserman metrics do not exist

All known examples corresponding to types II and III have nilpotent Jacobi operators.

Conjecture

The Jacobi operators of a Jordan-Osserman manifold are either diagonalizable or nilpotent.

NEW EXAMPLES OF OSSERMAN METRICS

Objective

To construct examples of Type II Osserman metrics whose Jacobi operators have nonzero eigenvalues.

$$\frac{1}{\langle X, X\rangle} R_X = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 1 & \beta \end{pmatrix}$$

$Ker(R_X - \beta Id)$

Lorentzian signature

Complex space forms	$\longleftrightarrow$	Paracomplex space forms

Para-Kähler Osserman 4-manifolds are either paracomplex space forms or Ricci flat.

$\downarrow$ Para-Kähler manifolds are Walker

Para-Hermitian structures on Walker manifolds

WALKER PARA-HERMITIAN STRUCTURES

For a Walker 4-manifold with coordinates $(x_1, \dots, x_4)$ and parallel null distribution $\mathcal{D} = \{\partial_1, \partial_2\}$, consider the almost para-Hermitian structure:

$$J\partial_1 = -\partial_1 \qquad J\partial_2 = \partial_2$$

$$\begin{matrix} J^2 = Id \\ g(J\cdot, J\cdot) = -g(\cdot,\cdot) \end{matrix}$$

$$J\partial_3 = -a\partial_1 + \partial_3 \qquad J\partial_4 = b\partial_2 - \partial_4$$

- J is integrable $\Leftrightarrow$ $a_2 = b_1 = 0$

- (g, J) is Einstein para-Hermitian if and only if

$$\left\{\begin{array}{l} \left[\begin{array}{l} a_{11} = b_{22}, \\ c_{11} = c_{22} = 0 \\ 2a_1c_2 - 2c_2^2 - 2ac_{12} + 4c_{23} = 0 \\ 2b_2c_1 - 2c_1^2 - 2bc_{12} + 4c_{14} = 0 \\ 2c_1c_2 - ca_{11} + 2a_{14} - cb_{22} \\ \qquad + 2b_{23} + 2cc_{12} - 2c_{13} - 2c_{24} = 0 \end{array}\right. \\ \left[\ a_2 = b_1 = 0 \right. \end{array}\right.$$

NEW EXAMPLES OF OSSERMAN METRICS

Type A $(\tau = 0)$

$$g : \begin{cases} a = x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b = x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c = x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4), \end{cases}$$

where

$$\begin{aligned} &PT - T^2 + 2T_3 = 0, \\ &QS - S^2 + 2S_4 = 0, \\ &ST + Q_3 - S_3 + P_4 - T_4 = 0. \end{aligned}$$

Type B $(\tau \neq 0)$

$$g : \begin{cases} a = x_1^2 \frac{\tau}{4} + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b = x_2^2 \frac{\tau}{4} + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c = \frac{2}{\tau} \{P_4(x_3, x_4) + Q_3(x_3, x_4)\}. \end{cases}$$

Type C $(\tau \neq 0)$

$$g : \begin{cases} a = x_1^2 \frac{\tau}{6} + x_1 P + \frac{6}{\tau} \left\{PT - T^2 + 2T_3\right\}, \\ b = x_2^2 \frac{\tau}{6} + x_2 Q + \frac{6}{\tau} \left\{QS - S^2 + 2S_4\right\}, \\ c = x_1 x_2 \frac{\tau}{6} + x_1 S + x_2 T \\ \qquad + \frac{6}{\tau} \{ST + Q_3 - S_3 + P_4 - T_4\}. \end{cases}$$

SOME PROPERTIES:

- → Type C metrics are Osserman with eigenvalues

$$\{\, 0,\ \frac{\tau}{6},\ \frac{\tau}{24},\ \frac{\tau}{24}\,\}.$$

  Jacobi operators are nondiagonalizable in an open subset where the metric is Jordan-Osserman.

- → Jacobi operators are diagonalizable iff metric corresponds to a paracomplex space form ($\nabla R = 0$).

- → First, second and third order scalar curvature invariants coincide with those of complex and paracomplex space forms.

- → Examples are null Osserman but never null Jordan-Osserman.

- → Examples are nonsymmetric although $\|\nabla R\| = 0$.

- → Examples are Szabó but not Jordan-Szabó. (Counterexamples at the algebraic level).

- → Examples are not 1-curvature homogeneous even if they are Jordan-Osserman.

GENERAL DESCRIPTION

Obtain a description of all Type II Osserman metrics metrics with non nilpotent Jacobi operators.

Fact A Type II Osserman $\longrightarrow \begin{cases} \alpha = \beta (= 0), \text{ or} \\ \alpha = 4\beta. \end{cases}$

Fact B Type II Osserman $\alpha = 4\beta$ $\longrightarrow$ Walker metric.

Fact C M^4 pointwise Osserman $\longleftrightarrow \begin{cases} \text{Einstein} \\ \text{self-dual (or ASD).} \end{cases}$

Fact D Jordan normal form of R_X $\overset{1:1}{\longleftrightarrow}$ Jordan normal form of $W^{\pm}$

$$R \equiv \frac{\tau}{12} Id_{\Lambda^2(V)} + \frac{1}{2} Ric_0 + \left(\begin{array}{c|c} W^+ & 0 \\ \hline 0 & W^- \end{array} \right)$$

Problem

Describe all self-dual (or anti-self-dual) Einstein Walker metrics

ANTI-SELF-DUAL WALKER METRICS

Choose an orthonormal basis as follows:

$$e_1 = \tfrac{1}{2}(1-a)\partial_1 + \partial_3, \qquad e_2 = -c\partial_1 + \tfrac{1}{2}(1-b)\partial_2 + \partial_4,$$
$$e_3 = -\tfrac{1}{2}(1+a)\partial_1 + \partial_3, \quad e_4 = -c\partial_1 - \tfrac{1}{2}(1+b)\partial_2 + \partial_4.$$

- Self-dual Weyl curvature tensor:

$$W^+ = \begin{pmatrix} W^+_{11} & W^+_{12} & W^+_{11} + \frac{\tau}{12} \\ -W^+_{12} & \frac{\tau}{6} & -W^+_{12} \\ -(W^+_{11} + \frac{\tau}{12}) & -W^+_{12} & -(W^+_{11} + \frac{\tau}{6}) \end{pmatrix}.$$

→ Eigenvalues of W^+ are $\left\{\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}\right\}$.

→ W^+ is diagonalizable if and only if

$$\tau^2 + 12\tau W^+_{11} + 48\left(W^+_{12}\right)^2 = 0.$$

Anti-self-dual Osserman Walker metrics are Ricci flat.
Nilpotent Jacobi operators

SELF-DUAL WALKER METRICS

- (M,g) is self-dual ($W^- = 0$) if and only if

$$\begin{cases} W^-_{11} = -\frac{1}{12}(a_{11} + 3a_{22} + 3b_{11} + b_{22} - 4c_{12}) = 0, \\ W^-_{22} = -\frac{1}{6}(a_{11} + b_{22} - 4c_{12}) = 0, \\ W^-_{33} = \frac{1}{12}(a_{11} - 3a_{22} - 3b_{11} + b_{22} - 4c_{12}) = 0, \\ W^-_{12} = \frac{1}{4}(a_{12} + b_{12} - c_{11} - c_{22}) = 0, \\ W^-_{13} = \frac{1}{4}(a_{22} - b_{11}) = 0, \\ W^-_{23} = -\frac{1}{4}(a_{12} - b_{12} + c_{11} - c_{22}) = 0, \end{cases}$$

Theorem

A Walker metric is self-dual if and only if the defining functions a, b and c are given by

$$a(x_1,x_2,x_3,x_4) = x_1^3\mathcal{A} + x_1^2\mathcal{B} + x_1^2x_2\mathcal{C} + x_1x_2\mathcal{D} + x_1P + x_2Q + \xi,$$

$$b(x_1,x_2,x_3,x_4) = x_2^3\mathcal{C} + x_2^2\mathcal{E} + x_1x_2^2\mathcal{A} + x_1x_2\mathcal{F} + x_1S + x_2T + \eta,$$

$$c(x_1,x_2,x_3,x_4) = \frac{1}{2}x_1^2\mathcal{F} + \frac{1}{2}x_2^2\mathcal{D} + x_1^2x_2\mathcal{A} + x_1x_2^2\mathcal{C} + \frac{1}{2}x_1x_2(\mathcal{B}+\mathcal{E}) + x_1U + x_2V + \gamma,$$

where capital, calligraphic and Greek letters are smooth functions of (x_3,x_4).

Theorem

Let (M, g) *be a four-dimensional* Type II *Osserman manifold. Then the Jacobi operators are nilpotent or otherwise there exist local coordinates* $(x_1, \ldots, x_4)$ *such that*

$$dx^1 dx^3 + dx^2 dx^4 + \sum_{i \le j = 3,4} s_{ij} dx^i dx^j$$

for some functions $s_{ij}(x_1, \ldots, x_4)$ *as follows*

$$\begin{aligned} s_{33} = & x_1^2 \tfrac{\tau}{6} + x_1 P + x_2 Q \\ & + \tfrac{6}{\tau}\{Q(T-U) + V(P-V) - 2(Q_4 - V_3)\}, \\ s_{44} = & x_2^2 \tfrac{\tau}{6} + x_1 S + x_2 T \\ & + \tfrac{6}{\tau}\{S(P-V) + U(T-U) - 2(S_3 - U_4)\}, \\ s_{34} = & x_1 x_2 \tfrac{\tau}{6} + x_1 U + x_2 V \\ & + \tfrac{6}{\tau}\{-QS + UV + T_3 - U_3 + P_4 - V_4\}, \end{aligned}$$

and arbitrary functions P, Q, S, T, U, V *depending only on* (x_3, x_4).

Jordan-Osserman Walker 4-manifold:

- → paracomplex space form
- → nilpotent Jacobi operators (2- or 3-step)
- → previous examples

OPEN PROBLEM: Type III Osserman 4-manifolds