Walker metrics: Applications to Osserman manifolds

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WALKER METRICS

Definition

\((M^n, g)\) is said to be a Walker manifold if it admits a degenerate parallel plane field.

Examples:

- Plane wave metrics
- Weakly-irreducible pseudo-Riemannian metrics
- Two-step nilpotent Lie g. with degenerate center
- Metrics on tangent and cotangent bundles
  - Complete lifts to \(TM\)
  - Riemann extensions to \(T^*M\)
- Para-Kähler and Hypersymplectic structures.

Underlying structure of some geometric problems:

- Nonuniqueness of the metric for the Levi Civita connection.
- Einstein hypersurfaces in indefinite space forms.
- Curvature conditions:
  - Self-dual K. and A.K. Einstein metrics,
  - Harmonic manifolds, Osserman metrics, …
CANONICAL FORM OF A WALKER METRIC

\((M^n, g)\) admits a degenerate parallel plane field \(D^r\) there are coordinates \((x_1, \ldots, x_n)\) such that

\[
(g_{ij}) = \begin{pmatrix}
0 & 0 & I_r & \\
0 & A & H & \\
I_r & tH & B_{r \times r} & \\
\end{pmatrix}, \quad A, B \text{ symmetric} \quad \quad A, H \text{ independent of } (x_1, \ldots, x_r).
\]

\(\checkmark\) \(D\) is spanned by \(\{\partial_1, \ldots, \partial_r\}\).

• Special cases:

\(n = 4, r = 2\)

\[
g(x_1, \ldots, x_4) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & a & c \\
0 & 1 & c & b
\end{pmatrix}, \quad a, b, c \text{ functions of } (x_1, \ldots, x_4).
\]

There exists two orthogonal parallel null vectors

\[
a \equiv 0, \quad c \equiv 0, \quad b \equiv b(x_3, x_4)
\]

\(\checkmark\) Any such metric globally defined on \(\mathbb{R}^4\) is geodesically complete.
OSSERMAN METRICS

\((M^n, g)\): pseudo-Riemannian manifold with curvature tensor \(R\). For a point \(p \in M\) the Jacobi operator is defined by

\[ R_X = R(X, \cdot)X, \quad X \in T_pM. \]

Definitions

- \((M, g)\) is **globally Osserman** if the eigenvalues of the Jacobi operators are constant on \(S^\pm(TM)\).

- \((M, g)\) is **pointwise Osserman** if the eigenvalues of \(R_X\) are independent of the direction \(X\), but may change from point to point.

Two-point homogeneous \(\rightarrow\) globally Osserman

- \((M, g)\) is **(pointwise) Jordan-Osserman** if the Jordan normal form of the Jacobi operators is (pointwise) constant on \(S^\pm(TM)\).

The eigenvalue structure does not determine the Jordan normal form in the indefinite setting.
OSSERMAN METRICS. AN OVERVIEW

### Riemannian case

- Riemannian Osserman spaces are locally two-point homogeneous \((\dim M \neq 16)\).
- Osserman algebraic curvature tensors exist which do not correspond to any symmetric space.

### Lorentzian case

- Osserman algebraic curvature tensors are of constant curvature.

### Higher signature

- Nonsymmetric and non locally homogeneous Osserman metrics exist.
- Partial classification is available under additional conditions:
  - \(\dim M = 4 \) + diagonalizable Jacobi operators
  - \(\dim M = 4 \) + \(\nabla R = 0\)
  - Diagonalizable Jacobi operators with exactly two-distinct eigenvalues
SOME ALGEBRAIC FACTS

- Spacelike and timelike Osserman conditions are equivalent at the algebraic level.

- Spacelike and timelike Jordan-Osserman conditions are not equivalent.

- The spacelike Jacobi operators of a spacelike Jordan-Osserman algebraic curvature tensor are necessarily diagonalizable whenever $p < q$ and the signature is $(- \cdots - , + \cdots +)$.

- Let $\mathcal{A}$ be a self-adjoint map of $\mathbb{R}^{(l,l)}$. There exist Jordan-Osserman algebraic curvature tensors in $\mathbb{R}^{(p,p)}$, with $p = 2^{l+1}$ such that

\[
R_X = \begin{pmatrix}
\langle X, X \rangle \mathcal{A} & 0 \\
0 & 0
\end{pmatrix}.
\]
FOUR-DIMENSIONAL OSSERMAN METRICS

Osserman $\rightarrow$ Einstein

\[
\begin{cases}
\text{Osserman} + \\
\text{dim} M = 3
\end{cases}
\rightarrow \text{constant sectional curvature}
\]

Special features of dimension four

• First nontrivial case

\[
\begin{cases}
\text{pointwise Osserman} \\
\text{Einstein self-dual (or anti-self-dual)}
\end{cases}
\leftrightarrow
\begin{cases}
\text{Einstein self-dual (or anti-self-dual)}
\end{cases}
\]

• All possible Jordan normal forms are realized at the algebraic level

\[
X^\perp \xrightarrow{R_X} X^\perp
\]

Type Ia:

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{pmatrix}
\]

Type Ib:

\[
\begin{pmatrix}
\alpha & \beta & 0 \\
-\beta & \alpha & 0 \\
0 & 0 & \gamma
\end{pmatrix}
\]

Type II:

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 1 & \beta
\end{pmatrix}
\]

Type III:

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
1 & \alpha & 0 \\
0 & 1 & \alpha
\end{pmatrix}
\]
FOUR-DIMENSIONAL OSSERMAN METRICS

Type Ia Osserman metrics are classified:

→ real space forms
→ complex space forms
→ paracomplex space forms

Type Ib Osserman metrics do not exist

All known examples corresponding to types II and III have nilpotent Jacobi operators.

Conjecture
The Jacobi operators of a Jordan-Osserman manifold are either diagonalizable or nilpotent.
NEW EXAMPLES OF OSSERMAN METRICS

Objective
To construct examples of Type II Osserman metrics whose Jacobi operators have nonzero eigenvalues.

\[
\frac{1}{\langle X, X \rangle} R_X = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 1 & \beta \end{pmatrix}
\]

\(Ker(R_X - \beta Id)\)

Lorentzian signature

Complex space forms \(\leftrightarrow\) Paracomplex space forms

Para-Kähler Osserman 4-manifolds are either paracomplex space forms or Ricci flat.

Para-Kähler manifolds are Walker

Para-Hermitian structures on Walker manifolds
WALKER PARA-HERMITIAN STRUCTURES

For a Walker 4-manifold with coordinates \((x_1, \ldots, x_4)\) and parallel null distribution \(\mathcal{D} = \{\partial_1, \partial_2\}\), consider the almost para-Hermitian structure:

\[
\begin{align*}
J\partial_1 &= -\partial_1 & J\partial_2 &= \partial_2 \\
J\partial_3 &= -a\partial_1 + \partial_3 & J\partial_4 &= b\partial_2 - \partial_4
\end{align*}
\]

\(J\) is integrable \(\iff a_2 = b_1 = 0\)

\((g, J)\) is Einstein para-Hermitian if and only if

\[
\begin{cases}
a_{11} = b_{22}, \\
c_{11} = c_{22} = 0 \\
2a_1c_2 - 2c_2^2 - 2ac_{12} + 4c_{23} = 0 \\
2b_2c_1 - 2c_1^2 - 2bc_{12} + 4c_{14} = 0 \\
2c_1c_2 - ca_{11} + 2a_{14} - cb_{22} \\
\phantom{2c_1c_2} + 2b_{23} + 2cc_{12} - 2c_{13} - 2c_{24} = 0 \\
a_2 = b_1 = 0
\end{cases}
\]
NEW EXAMPLES OF OSSERMAN METRICS

**Type A** ($\tau = 0$)

$$
g : \begin{cases} 
a &= x_1 P(x_3, x_4) + \xi(x_3, x_4), \\
b &= x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\
c &= x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4), 
\end{cases}
$$

where

$$
PT - T^2 + 2T_3 = 0, \\
QS - S^2 + 2S_4 = 0, \\
ST + Q_3 - S_3 + P_4 - T_4 = 0.
$$

**Type B** ($\tau \neq 0$)

$$
g : \begin{cases} 
a &= x_1^{1/4} + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\
b &= x_2^{1/4} + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\
c &= \frac{2}{\tau} \{ P_4(x_3, x_4) + Q_3(x_3, x_4) \}.
\end{cases}
$$

**Type C** ($\tau \neq 0$)

$$
g : \begin{cases} 
a &= x_1^{1/6} + x_1 P + \frac{6}{\tau} \{ PT - T^2 + 2T_3 \}, \\
b &= x_2^{1/6} + x_2 Q + \frac{6}{\tau} \{ QS - S^2 + 2S_4 \}, \\
c &= x_1 x_2^{1/6} + x_1 S + x_2 T + \frac{6}{\tau} \{ ST + Q_3 - S_3 + P_4 - T_4 \}.
\end{cases}
$$
SOME PROPERTIES:

→ **Type C** metrics are Osserman with eigenvalues
\[
\{ 0, \frac{\tau}{6}, \frac{\tau}{24}, \frac{\tau}{24} \}.
\]

Jacobi operators are nondiagonalizable in an open subset where the metric is Jordan-Osserman.

→ Jacobi operators are diagonalizable iff metric corresponds to a paracomplex space form \((\nabla R = 0)\).

→ First, second and third order scalar curvature invariants coincide with those of complex and paracomplex space forms.

→ Examples are null Osserman but never null Jordan-Osserman.

→ Examples are nonsymmetric although \(|\nabla R| = 0\).

→ Examples are Szabó but not Jordan-Szabó. (Counterexamples at the algebraic level).

→ Examples are not 1-curvature homogeneous even if they are Jordan-Osserman.
Obtain a description of all Type II Osserman metrics with non nilpotent Jacobi operators.

Fact A

Type II Osserman $\rightarrow$ \begin{cases} \alpha = \beta (= 0), \\
\alpha = 4\beta. \end{cases}

Fact B

Type II Osserman $\alpha = 4\beta \rightarrow$ Walker metric.

Fact C

$M^4$ pointwise Osserman $\leftrightarrow$ \begin{cases} \text{Einstein self-dual (or ASD).} \end{cases}

Fact D

Jordan normal form of $R_X \leftrightarrow$ Jordan normal form of $W^\pm$

$$R \equiv \frac{\tau}{12} \text{Id} \wedge^2(V) + \frac{1}{2} \text{Ric}_0 + \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix}$$

Problem

Describe all self-dual (or anti-self-dual) Einstein Walker metrics
ANTI-SELF-DUAL WALKER METRICS

Choose an orthonormal basis as follows:

\[ e_1 = \frac{1}{2}(1 - a) \partial_1 + \partial_3, \quad e_2 = -c \partial_1 + \frac{1}{2}(1 - b) \partial_2 + \partial_4, \]
\[ e_3 = -\frac{1}{2}(1 + a) \partial_1 + \partial_3, \quad e_4 = -c \partial_1 - \frac{1}{2}(1 + b) \partial_2 + \partial_4. \]

- Self-dual Weyl curvature tensor:

\[ W^+ = \begin{pmatrix} W^+_{11} & W^+_{12} & W^+_{11} + \frac{\tau}{12} \\
-W^+_{12} & \frac{\tau}{6} & -W^+_{12} \\
-(W^+_{11} + \frac{\tau}{12}) & -W^+_{12} & -(W^+_{11} + \frac{\tau}{6}) \end{pmatrix}. \]

→ Eigenvalues of \( W^+ \) are \( \left\{ \frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12} \right\} \).

→ \( W^+ \) is diagonalizable if and only if

\[ \tau^2 + 12\tau W^+_{11} + 48 (W^+_{12})^2 = 0. \]

Anti-self-dual Osserman Walker metrics are Ricci flat.

Nilpotent Jacobi operators
**SELF-DUAL WALKER METRICS**

- \((M, g)\) is self-dual \((W^- = 0)\) if and only if

\[
\begin{align*}
W^-_{11} &= -\frac{1}{12}(a_{11} + 3a_{22} + 3b_{11} + b_{22} - 4c_{12}) = 0, \\
W^-_{22} &= -\frac{1}{6}(a_{11} + b_{22} - 4c_{12}) = 0, \\
W^-_{33} &= \frac{1}{12}(a_{11} - 3a_{22} - 3b_{11} + b_{22} - 4c_{12}) = 0, \\
W^-_{12} &= \frac{1}{4}(a_{12} + b_{12} - c_{11} - c_{22}) = 0, \\
W^-_{13} &= \frac{1}{4}(a_{22} - b_{11}) = 0, \\
W^-_{23} &= -\frac{1}{4}(a_{12} - b_{12} + c_{11} - c_{22}) = 0,
\end{align*}
\]

**Theorem**

A Walker metric is self-dual if and only if the defining functions \(a, b\) and \(c\) are given by

\[
\begin{align*}
a(x_1, x_2, x_3, x_4) &= x_1^3A + x_1^2B + x_1x_2C + x_1x_2D + x_1P + x_2Q + \xi, \\
b(x_1, x_2, x_3, x_4) &= x_2^3C + x_2^2E + x_1x_2^2A + x_1x_2F + x_1S + x_2T + \eta, \\
c(x_1, x_2, x_3, x_4) &= \frac{1}{2}x_1^2F + \frac{1}{2}x_2^2D + x_1x_2A + x_1x_2^2C + \frac{1}{2}x_1x_2(B + \varepsilon) \\
&\quad + x_1U + x_2V + \gamma,
\end{align*}
\]

where capital, calligraphic and Greek letters are smooth functions of \((x_3, x_4)\).
Theorem

Let \((M, g)\) be a four-dimensional Type II Osserman manifold. Then the Jacobi operators are nilpotent or otherwise there exist local coordinates \((x_1, \ldots, x_4)\) such that

\[
dx^1 dx^3 + dx^2 dx^4 + \sum_{i\leq j=3,4} s_{ij} dx^i dx^j
\]

for some functions \(s_{ij}(x_1, \ldots, x_4)\) as follows

\[
s_{33} = x_1^2 \frac{\tau}{16} + x_1 P + x_2 Q + \frac{6}{\tau} \{Q(T-U) + V(P-V) - 2(Q_4-V_3)\},
\]

\[
s_{44} = x_1^2 \frac{\tau}{26} + x_1 S + x_2 T + \frac{6}{\tau} \{S(P-V) + U(T-U) - 2(S_3-U_4)\},
\]

\[
s_{34} = x_1 x_2 \frac{\tau}{6} + x_1 U + x_2 V + \frac{6}{\tau} \{- QS + UV + T_3 - U_3 + P_4 - V_4\},
\]

and arbitrary functions \(P, Q, S, T, U, V\) depending only on \((x_3, x_4)\).

Jordan-Osserman Walker 4-manifold:

\[\rightarrow\] paracomplex space form
\[\rightarrow\] nilpotent Jacobi operators (2- or 3-step)
\[\rightarrow\] previous examples

OPEN PROBLEM: Type III Osserman 4-manifolds