A new development of the GKP construction

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BOUNDARIES OF SPACETIMES

- Interest:

  - Singularity Theory.

  - Asymptotic properties of fields and gravitation.

    ⋮

  - Quantum aspects of gravity.

- Many different boundaries in Relativity:

Conformal boundary, g-boundary, b-boundary, Meyer's construction, a-boundary, isocausal boundary...*causal boundary*.

CAUSAL BOUNDARY

• Motivation: Find a "systematic" and "intrinsic" procedure to obtain a "natural" and "unique" boundary for "general" spacetimes.

• Guide Properties:

(1) Any inextensible causal curve must have some limit in the boundary.

(2) Boundary exclusively based on the global causal structure of the spacetime.

(3) Extendibility of the causality and topology of the spacetime to the attached boundary.

THE GKP APPROACH '72

- Definition of *Future Causal Boundary* $\partial^+(V)$:

(i) Attach a *future ideal point* to every inextensible $\uparrow$-timelike curve γ,

(ii) two different curves $\gamma \neq \gamma'$ are attached the same ideal point iff they have the same past $I^-[\gamma] = I^-[\gamma']$.

Then,

$\partial^+(V) :=$ set of all future ideal points.

- Definition of *Future Causal Completion* V^+:

$$V^+ := V \cup \partial^+(V)$$
$$= \{\text{real points}\} \cup \{\text{future ideal points}\}.$$

• Equivalent re-formulation of the Future Causal Boundary:

- *Past set*: $P \subset V$ such that $I^{-}[P] = P$.

- *Indecomposable past set (IP)*: past set P which cannot be expressed as union of two proper past sets.

- *Proper IP (PIP)*: IP P such that $P = I^{-}(p)$.

- *Terminal IP (TIP)*: IP P such that $P \neq I^{-}(p)$ for any $p \in V$.

Then, we have the following identifications:

$$V \equiv \text{PIPs}, \qquad \partial^{+}(V) \equiv \text{TIPs},$$

$$\begin{aligned} V^{+} &:= V \cup \partial^{+}(V) \\ &\equiv \text{PIPs} \cup \text{TIPs} = \text{IPs}. \end{aligned}$$

• The dual notions of previous definitions are introduced analogously: *past ideal point*, *future set*, *IF*, *PIF*, *TIF*.

• This provides the following constructions for the past:

- *Past Causal Boundary* $\partial^{-}(V)$;

$$\begin{aligned} \partial^{-}(V) \; &:= \text{set of all past ideal points} \\ &\equiv \text{TIFs.} \end{aligned}$$

- *Past Causal Completion* V^{-};

$$\begin{aligned} V^{-} \; &:= V \cup \partial^{-}(V) \\ &= \{\text{real points}\} \cup \{\text{past ideal points}\} \\ &\equiv \text{PIFs} \cup \text{TIFs} = \text{IFs.} \end{aligned}$$

EXTENDED CAUSAL RELATIONS

- Causal Relations for the Future Causal Completion:

  - *Causal relation*: $P \prec P'$ iff $P \subset P'$.

  - *Chronological relation*: $P \ll P'$ iff there exists $r \in P'$ such that $P \subset I^{-}(r)$.

- Causal Relations for the Past Causal Completion:

  - *Causal relation*: $F \prec F'$ iff $F' \subset F$.

  - *Chronological relation*: $F \ll F'$ iff there exists $r \in F$ such that $F' \subset I^{+}(r)$.

Remark: May include additional $\ll$ relations.

(TOTAL) CAUSAL BOUNDARY

• Naive approach: a priori, the most natural definition for the total causal boundary is

$$\partial(V) := \partial^+(V) \cup \partial^-(V).$$

• Problem: this definition DOES NOT WORK in general! Some NON-TRIVIAL identifications may be needed!

• First, they construct a pre-completion $V^\sharp$ where obvious identifications between PIPs and PIFs are established:

$$V^\sharp := V^+ \cup V^- / \sim$$

where $I^-(p) \sim I^+(p)$ for all $p \in V$.

IDENTIFICATIONS AND TOPOLOGY

- *Alexandrov Topology* on V: subbase formed by $I^{-}(p)$, $I^{+}(p)$, $V \setminus \overline{I^{-}(p)}$, $V \setminus \overline{I^{+}(p)}$.

- *Generalized Alexandrov Topology* on $V^{\sharp}$: subbase formed by

$$
\begin{aligned}
F^{int} &:= \{P \in V^{+} : P \cap F \neq \emptyset\} \\
F^{ext} &:= \{P \in V^{+} : P = I^{-}[S] \Rightarrow I^{+}[S] \not\subset F\} \\
P^{int} &:= \{F \in V^{-} : P \cap F \neq \emptyset\} \\
P^{ext} &:= \{F \in V^{-} : F = I^{+}[S] \Rightarrow I^{-}[S] \not\subset P\}.
\end{aligned}
$$

- Definition: The *GKP causal completion* is the quotient space

$$\overline{V} := V^{\sharp}/\sim_h$$

$\sim_h :=$ minimum equivalence relation $\sim$ such that $V^{\sharp}/\sim$ is Hausdorff.

- Definition: The *GKP causal boundary* is

$$\partial(V) := \overline{V} \setminus V.$$

• Benefits: “Intrinsic” and “systematic” procedure. Provides an “unique” and “general” boundary.

• Objections:

- The singularity of Taub spacetime

$$ds^2 = z^{-1/2}(dt^2 - dz^2) - z(dx^2 + dy^2), \quad z > 0$$

becomes an unique point! [KLL’86].

- The causal structure of the (total) boundary is not analyzed.

- The topology for the completion of Minkowski does not coincide with that derived from the embedding into ESU [H’00].

- This method actually needs *stably causal* spacetimes [S’88].

- Causes:

  - The treatment of past and future boundaries separately is an artificial procedure.

  - Hausdorffness condition seems to be incompatible with the causal boundary approach.

OTHER DEVELOPMENTS OF THE GKP APPROACH

BUDIC-SACHS '74:

- Identifications directly defined on $V^+ \cup V^-$:

$$P \sim_{bs} F \quad \text{iff} \quad P = \downarrow F \quad \text{or} \quad F = \uparrow P.$$

- A new extended causal structure is defined:

$$\begin{array}{lll} P \prec P' & \text{iff} & P \subset P' \\ P \ll P' & \text{iff} & P' \cap (\uparrow P) \neq \emptyset \\ P \prec F & \text{iff} & \exists\ \hat{L},\ \check{L}\ \ s.t.\ \ P \subset \hat{L},\ F \subset \check{L} \\ P \ll F & \text{iff} & (\downarrow F) \cap (\uparrow P) \neq \emptyset \\ \vdots & & \end{array}$$

- New generalized Alexandrov topology, now defined directly on $\overline{V}$: subbase formed by $I^+(p)$, $I^-(p)$, $\overline{V} \setminus J^-(p)$, $\overline{V} \setminus J^+(p)$.

• Benefits:

Satisfactory extension of $\ll$ to $\overline{V}$ without additional $\ll$ relations on V.

Satisfactory extension of $\prec$ to $\overline{V}$ without additional $\prec$ relations on V iff V is *causally simple*.

If V is *causally continuous*, the resulting quotient topology is Hausdorff and V becomes topologically and densely embedded into $\overline{V}$.

• Objections:

Very restrictive construction. Only applicable to causally continuous spacetimes.

Bad topological behaviors in some examples [KL'88].

RACZ '87:

- ⋆ Generalized Alexandrov topology defined on $V^{+} \cup V^{-}$: subbase formed by F^{int}, F^{ext}, P^{int}, P^{ext}, with

$$F^{int} := \{A \in V^{+} \cup V^{-} : A \in V^{+}, A \cap F \neq \emptyset \text{ or } A \in V^{-}, I^{+}[S] = A \Rightarrow I^{-}[S] \cap F \neq \emptyset\}$$

$$F^{ext} := \{A \in V^{+} \cup V^{-} : A \in V^{-}, A \not\subset F \text{ or } A \in V^{+}, I^{-}[S] = A \Rightarrow I^{+}[S] \not\subset F\}.$$

- ⋆ Consider the minimum set of identifications $\sim_r$ which ensures $I^{-}(p) \sim_r I^{+}(p)$.

- ⋆ Under certain technical conditions on V the resulting quotient topology is Hausdorff.

- ⋆ Provide specific construction for *stably causal* spacetimes.

• Benefits:

Reproduce the 1-dimensional character of the singularity region of Taub spacetime.

• Objections:

Essentially the same as the GKP approach.

Bad topological behaviors in some examples [KL'92].

SZABADOS '88 '89

- $\star$ Identifications $\sim_s$ directly defined on $V^\sharp$:

$$P \sim_s F \text{ iff } \begin{cases} P & \text{maximal IP into } \downarrow F \\ F & \text{maximal IF into } \uparrow P \end{cases}$$

  Other relations $P \sim_s P'$, $F \sim_s F'$ are also introduced.

- $\star$ Chronological relation: $m \ll m'$ iff for some $F_\alpha \in \pi^{-1}(m)$ and some $P'_\mu \in \pi^{-1}(m')$, it is $F_\alpha \cap P'_\mu \neq \emptyset$.

- $\star$ Causal relation: $m \prec m'$ iff $I^+(m) \supset I^+(m')$ and $I^-(m) \subset I^-(m')$.

- $\star$ They take the quotient of the GKP generalized Alexandrov topology on $V^\sharp$.

- Benefits:

Overcome most of the troubles of the GKP approach.

The resulting topology is Hausdorff.

- Objections:

Appear spurious $\ll$ relations inherent to the Szabados identification rule [MR'03].

Bad topological limits in concrete examples are shown in [KL'92], [MR'03].

MAROLF-ROSS '03

- ⋆ Relation $\sim_s$ is used to form pairs, instead of establishing identifications between indecomposable sets:

$$(P,F) \in \overline{V} \text{ iff } \begin{cases} P \sim_s F \\ P = \emptyset,\ F \not\sim_s P' \ \forall P' \in V^+ \\ F = \emptyset,\ P \not\sim_s F' \ \forall F' \in V^-. \end{cases}$$

- ⋆ They essentially adopt the Szabados' chronology:

$$(P,F) \ll (P',F') \quad \text{iff} \quad F \cap P' \neq \emptyset.$$

- ⋆ Rather technical topology: defined by imposing $\overline{V} \setminus L^{\pm}(\overline{S})$, $\overline{S} \subset \overline{V}$ to be open, with $L^{\pm}$ operators entirely based on the chronology of V.

- Benefits:

This chronology does not introduce spurious relations.

Topology with many satisfactory properties: V becomes topologically embedded into $\overline{V}$, the boundary $\partial(V)$ is closed in $\overline{V}$...

- Objections:

Topology with too many convergent sequences. Bad separation properties: it is not T_1!

Another alternative topology suggested by Marolf-Ross behaves even worse!

THE CHRONOLOGICAL BOUNDARY

• Guide Properties:

(1) Any inextensible timelike curve must have some limit in the boundary.

(2) Boundary exclusively based on the “global” chronological structure of the spacetime.

(3) Extendibility of the chronology and topology of the spacetime to the attached boundary.

• From (2), this construction is only applicable to (past/future) distinguishing spacetimes.

CONSTRUCTION

⋆ Idea: The completion $\overline{V}$ will be formed by all "endpoints" of chains (timelike curves) in V.

• Definition: A pair $(P, F) \in V_p \times V_f$ is said *generated* by a chain (timelike curve) $\delta \subset V$ if its components are the "limits" of the pasts and the futures of the points of δ; that is

$$P = I^{-}(LI(\{I^{-}(p_n)\}))$$
$$F = I^{+}(LI(\{I^{+}(p_n)\})), \quad \delta = \{p_n\},$$

where $LI(A_n) := \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$.

• Definition: A pair $(P, F) \in V_p \times V_f$ is said *elemental* if there is no another pair (P', F') generated by some chain such that

$$\text{dec}(P') \subset \text{dec}(P), \quad \text{dec}(F') \subset \text{dec}(F),$$

where

$$\text{dec}(P) := \{P_\alpha\}, \ P_\alpha \text{ maximal IP inside } P,$$
$$\text{dec}(F) := \{F_\alpha\}, \ F_\alpha \text{ maximal IF inside } F.$$

• Definition: An elemental pair $(P, F) \in V_p \times V_f$ is the *endpoint* of a chain $\delta \subset V$ if it is generated by δ.

• Definition: The *chronological completion* $\overline{V}$ is the set of endpoints of all chains in V:

$$\overline{V} := \text{set of all endpoints.}$$

• Definition: The *chronological boundary* $\partial(V)$ is then

$$\partial(V) := \overline{V} \setminus V.$$

Properties:

-There exist chains "without" endpoints.

-The endpoint of a chain, if exists, is "unique". It is preserved by subsequences.

-The unique pair in $\overline{V}$ with some component equal to $I^-(p)$ or $I^+(p)$ is $(I^-(p), I^+(p))$.

CHRONOLOGY

- Definition: $(P, F) \ll (P', F')$ iff $F \cap P' \neq \emptyset$.

- Properties:

  - No spurious $\ll$ relations are introduced in V.

  - V is chronologically dense in $\overline{V}$.

  - $I^{-}((P, F)) \cap V = P, \quad I^{+}((P, F)) \cap V = F.$

$\star$ Our completion is applicable to more general objects than that of spacetime: the *chronological sets*.

- Theorem: Completing the completion gives nothing new: $\overline{\overline{V}} \cong \overline{V}$.

- Theorem: $\overline{V}$ is universal in a categorical sense.

CHRONOLOGICAL TOPOLOGY

- *Future limit-operator* [H'00]: $P \in \hat{L}(\sigma)$, $\sigma = \{P_n\}$ iff *(i)* $P \subset LI(\{P_n\})$ and *(ii)* P is maximal IP in $LS(\{P_n\})$, where

$$LS(\{P_n\}) := \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} P_k.$$

- *Past limit-operator*: $F \in \check{L}(\sigma)$, $\sigma = \{F_n\}$ iff *(i)* $F \subset LI(\{F_n\})$, *(ii)* F is maximal IF in $LS(\{F_n\})$.

- *Limit-operator*: Given a pair $(P, F) \in \overline{V}$ and a sequence σ in $\overline{V}$, we say $(P, F) \in L(\sigma)$ iff

$$\text{dec}(P) \subset \hat{L}(\sigma), \qquad \text{dec}(F) \subset \check{L}(\sigma).$$

- Definition: The *closed sets* of $\overline{V}$ with the *chronological topology* are those subsets $C \subset \overline{V}$ such that $L(\sigma) \subset C$ for any sequence $\sigma \subset C$.

Properties:

- Every chain $\delta \subset V$ has some limit in $\overline{V}$. If δ has endpoint then it is the "unique" limit.

- V is densely and topologically embedded into $\overline{V}$.

- $\partial(V)$ is always closed in $\overline{V}$.

- The chronological topology is always T_1.

- If two elements of $\overline{V}$ are non-Hausdorff related then they are necessarily in $\partial(V)$.

- Specially satisfactory limit behaviors in some examples.

$\star$ Global hyperbolicity is also characterized in terms of the chronological boundary.

M p-WAVES

$$V = M \times \mathbb{R}^2, \quad \langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle + 2dudv + H(x,u)du^2$$

$(M, \langle\cdot,\cdot\rangle)$ "arbitrary" Riemannian manifold
(v, u) natural coordinates of $\mathbb{R}^2$
$H : M \times \mathbb{R} \to \mathbb{R}$ "arbitrary" function ($\not\equiv 0$).

- [BN'02] The conformal boundary of maximally supersymmetric 10-dimensional plane wave

$$M = \mathbb{R}^8, \quad H(x,u) = -\sum_i \mu^2 x^i x^i,$$

  is a null "line" which spirals around ESU.

- [MR'03] The 1-dimensional character of the boundary also holds for more general plane waves ($-H$ quadratic). Now, the "causal boundary" is needed!!

⋆ The low dimensionality of the causal boundary suggests that causality "degenerates" asymptotically.

⋆ This seems to imply a "critical" behavior of the causality with respect to some metric coefficients.

In fact, in [-S'03] the causality of Mp-waves is shown to be critical w.r.t. a quadratic spatial growth of coefficient $-H$:

- Mp-waves are strongly causal if $-H$ is at most quadratic (i.e., plane waves).

- Mp-waves are globally hyperbolic if $-H$ is subquadratic (and M complete).

- Mp-waves are non-distinguishing if $-H$ is superquadratic.

CONCLUSIONS

- Our approach is entirely based on the global chronological structure of the spacetime.

- In particular, it provides an intrinsic and systematic method to construct an unique and natural boundary for any distinguishing spacetime.

- This boundary seems specially useful to study the "global" or "asymptotic" behavior of the causal structure of the spacetime.

- However, it is probably useless to study other aspects which require a deeper information from the spacetime, such as singularities!!