Multidimensional cosmological models with vanishing Weyl curvature tensor

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with decomposed metric

$$g = g^{(0)} + \sum_{i=1}^{n} e^{2\beta^i(x)} g^{(i)}$$

where x are some coordinates on the D_0-dimensional external spacetime M_0 and $g^{(i)}$ are the metrics on the internal spaces M_i $(i = 1, \ldots, n)$.

Model and general setup

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equipped with the metric

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where $f_1, \ldots, f_n : M_0 \longrightarrow \mathbb{R}$ are positive functions.

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- f_i: *scaling functions*

Local conformal flatness

Definition

A manifold (M, g) is *locally conformally flat* if for each point $p \in M$ there exists a neighbourhood $V \subset M$ and a diffeomorphism $\varphi : U \subset \mathbb{R}^n_s \longrightarrow V$, such that $\varphi^* g = \phi^2 g_{\mathbb{R}^n_s}$.

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  (C: *Schouten tensor*)
- $dim\, M \geq 4 \quad \rightarrow \quad W = 0$
  (W: *Weyl tensor*)

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Aim

To describe the local structure of locally conformally flat MCMs.

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Aim

To describe the local structure of locally conformally flat MCMs.

As a consequence, we will show some restrictions on the number and the geometry of the possible internal spaces of the model.

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Consequences

$M_0 \times_{f_1} M_1 \times \cdots \times_{f_n} M_n$: locally conformally flat MCM

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(i) $(M_0, g^{(0)})$ is locally conformally flat.

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(i) $(M_0, g^{(0)})$ is locally conformally flat.

(ii) $(M_i, g^{(i)})$ is a space of constant sectional curvature for all $i = 1, \ldots n$, provided that $\dim M_i = D_i \geq 2$.

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(iii) $M_0 \times_{f_1} M_1 \times \cdots \times \widehat{M_i} \times \cdots \times_{f_n} M_n$ is locally conformally flat for all $i = 1, \ldots, n$.

Some notation

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τ	scalar curvature
∇f	gradient of f
Δf	Laplacian of f
h_f	Hessian tensor of f
H_f	Hessian form of f

Weyl curvature tensor

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$$W(\alpha,\beta,\gamma,\delta) = R(\alpha,\beta,\gamma,\delta)$$

$$+ \tfrac{\tau}{(D-1)(D-2)}\{\langle\alpha,\gamma\rangle\langle\beta,\delta\rangle - \langle\beta,\gamma\rangle\langle\alpha,\delta\rangle\}$$

$$- \tfrac{1}{D-2}\{\rho(\alpha,\gamma)\langle\beta,\delta\rangle - \rho(\beta,\gamma)\langle\alpha,\delta\rangle + \langle\alpha,\gamma\rangle\rho(\beta,\delta) - \langle\beta,\gamma\rangle\rho(\alpha,\delta)\}$$

$(\alpha,\beta,\gamma,\delta \in \mathfrak{L}(M))$

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Weyl curvature tensor $(X, Y, Z \in \mathfrak{L}(M_0),\ U_a, V_a, T_a \in \mathfrak{L}(M_a))$

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$$R_{XY}Z = R^{M_0}_{XY}Z,$$

$$R_{V_aX}Y = \tfrac{1}{f_a}H_{f_a}(X,Y)V_a, \qquad R_{XU_a}V_a = \tfrac{\langle U_a, V_a\rangle}{f_a}h_{f_a}(X),$$

$$R_{U_bU_a}V_a = \tfrac{\langle U_a, V_a\rangle}{f_af_b}\langle\nabla f_a, \nabla f_b\rangle U_b, \qquad a \neq b,$$

$$R_{U_aV_a}T_a = R^{M_a}_{U_aV_a}T_a - \tfrac{\langle\nabla f_a, \nabla f_a\rangle}{f_a^2}\{\langle U_a, T_a\rangle V_a - \langle V_a, T_a\rangle U_a\}.$$

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Scalar curvature

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Scalar curvature

$$\tau = \tau^{M_0} + \sum_i \frac{1}{f_i^2}\tau^{M_i} - \sum_i D_i(D_i - 1)\frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} - 2\sum_i D_i \frac{\Delta f_i}{f_i} - \sum_{i,j\neq i} D_i D_j \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j}$$

First step: M_0 is a space of constant sectional curvature

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M_0 of constant sectional curvature

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$$g^{(0)} \mapsto \frac{1}{f_i^2} g^{(0)}$$ preserves the Einstein property

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$$W(X,Y,X,Y) \underset{X,Y \in \mathfrak{L}(M_0)}{=} \frac{D_i(D_i-1)\varepsilon_X \varepsilon_Y}{(D_0+D_i-1)(D_0+D_i-2)} \left\{ K^{M_0} + \frac{2}{D_0}\frac{\Delta f_i}{f_i} + \frac{K^{M_i}}{f_i^2} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} \right\}$$

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$$H_{f_i} = \frac{\Delta f_i}{D_0} g^{(0)}$$

$M_0 \times_{f_i} M_i$ is locally conformally flat

$$K^{M_0} + \frac{2}{D_0}\frac{\Delta f_i}{f_i} + \frac{K^{M_i}}{f_i^2} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} = 0 \qquad (\dim M_i \geq 2)$$

$M_0 \times_{f_i} M_i \times_{f_j} M_j$, $i \neq j$, is locally conformally flat

$$K^{M_0} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} = 0$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \quad \times_{f_1} \quad M_1 \quad \times \quad \cdots \quad \times_{f_a} \quad M_a \quad \times \quad \cdots \quad \times_{f_n} \quad M_n$$

First step: M_0 is a space of constant sectional curvature

$$\begin{array}{ccccccccc} M_0 & \times_{f_1} & M_1 & \times & \cdots & \times_{f_a} & M_a & \times & \cdots & \times_{f_n} & M_n \\ X, Y, \dots & & U_1, V_1, \dots & & & & U_a, V_a, \dots & & & & U_n, V_n, \dots \end{array}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \ldots$ $U_1, V_1, \ldots$ $U_a, V_a, \ldots$ $U_n, V_n, \ldots$

$$W(X,Y,X,Y) \qquad W(X,U_a,X,U_a)$$

$$W(U_a,U_b,U_a,U_b) \qquad W(U_a,V_a,U_a,V_a)$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \dots$ $U_1, V_1, \dots$ $U_a, V_a, \dots$ $U_n, V_n, \dots$

$W(X, Y, X, Y)$ $W(X, U_a, X, U_a)$

$W(U_a, U_b, U_a, U_b)$ $W(U_a, V_a, U_a, V_a)$

$$\varepsilon_X \varepsilon_Y W(X, Y, X, Y)$$

$$= \sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\}$$

$$+ \sum_{i,j\neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \ldots \qquad U_1, V_1, \ldots \qquad U_a, V_a, \ldots \qquad U_n, V_n, \ldots$

$W(X, Y, X, Y) \qquad W(X, U_a, X, U_a)$

$W(U_a, U_b, U_a, U_b) \qquad W(U_a, V_a, U_a, V_a)$

$$\varepsilon_X \varepsilon_Y W(X, Y, X, Y)$$

$$= \sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\}$$

$$+ \sum_{i, j \neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \quad \times_{f_1} \quad M_1 \quad \times \quad \cdots \quad \times_{f_a} \quad M_a \quad \times \quad \cdots \quad \times_{f_n} \quad M_n$$

$X, Y, \dots$ $\quad U_1, V_1, \dots$ $\quad U_a, V_a, \dots$ $\quad U_n, V_n, \dots$

$$W(X,Y,X,Y) = 0 \qquad W(X,U_a,X,U_a)$$

$$W(U_a,U_b,U_a,U_b) \qquad W(U_a,V_a,U_a,V_a)$$

First step: M_0 is a space of constant sectional curvature

$$\begin{array}{ccccccccccc} M_0 & \times_{f_1} & M_1 & \times & \cdots & \times_{f_a} & M_a & \times & \cdots & \times_{f_n} & M_n \\ X, Y, \dots & & U_1, V_1, \dots & & & & U_a, V_a, \dots & & & & U_n, V_n, \dots \end{array}$$

$$W(X, Y, X, Y) = 0 \qquad W(X, U_a, X, U_a)$$

$$W(U_a, U_b, U_a, U_b) \qquad W(U_a, V_a, U_a, V_a)$$

$$\begin{aligned} &\varepsilon_X \varepsilon_{U_a} W(X, U_a, X, U_a) \\ &\quad = \sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\} \\ &\quad + \sum_i \sum_{j \neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\} \\ &\quad + \sum_{i \neq a} \frac{D_i}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_i \rangle}{f_a f_i} - \frac{\Delta f_a}{D_0 f_a} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\} \\ &\quad + \frac{D_a - 1}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} - \frac{K^{M_a}}{f_a^2} - \frac{2\Delta f_a}{D_0 f_a} - K^{M_0} \right\} \end{aligned}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \dots$ $U_1, V_1, \dots$ $U_a, V_a, \dots$ $U_n, V_n, \dots$

$$W(X,Y,X,Y)=0 \qquad W(X,U_a,X,U_a)$$

$$W(U_a,U_b,U_a,U_b) \qquad W(U_a,V_a,U_a,V_a)$$

$$\begin{aligned}
&\varepsilon_X \varepsilon_{U_a} W(X,U_a,X,U_a)\\
&= \sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\}\\
&+ \sum_i \sum_{j\neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\}\\
&+ \sum_{i\neq a} \frac{D_i}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{D_0 f_a} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\}\\
&+ \frac{D_a-1}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_a\rangle}{f_a^2} - \frac{K^{M_a}}{f_a^2} - \frac{2\Delta f_a}{D_0 f_a} - K^{M_0} \right\}
\end{aligned}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \ldots$ $U_1, V_1, \ldots$ $U_a, V_a, \ldots$ $U_n, V_n, \ldots$

$W(X, Y, X, Y) = 0$ $W(X, U_a, X, U_a) = 0$

$W(U_a, U_b, U_a, U_b)$ $W(U_a, V_a, U_a, V_a)$

First step: M_0 is a space of constant sectional curvature

$$
\begin{array}{ccccccccccc}
M_0 & \times_{f_1} & M_1 & \times & \cdots & \times_{f_a} & M_a & \times & \cdots & \times_{f_n} & M_n \\
X, Y, \dots & & U_1, V_1, \dots & & & & U_a, V_a, \dots & & & & U_n, V_n, \dots
\end{array}
$$

$$W(X, Y, X, Y) = 0 \qquad W(X, U_a, X, U_a) = 0$$

$$W(U_a, U_b, U_a, U_b) \qquad W(U_a, V_a, U_a, V_a)$$

$$
\begin{aligned}
\varepsilon_{U_a}\varepsilon_{U_b} W(U_a, U_b, U_a, U_b) = {} & \textstyle\sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\} \\
& + \textstyle\sum_i \sum_{j \neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\} \\
& + \textstyle\sum_{i \neq a} \frac{D_i}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_i \rangle}{f_a f_i} - \frac{\Delta f_a}{D_0 f_a} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\} \\
& + \textstyle\sum_{i \neq b} \frac{D_i}{D-2} \left\{ \frac{\langle \nabla f_b, \nabla f_i \rangle}{f_b f_i} - \frac{\Delta f_b}{D_0 f_b} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\} \\
& + \frac{D_a - 1}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} - \frac{K^{M_a}}{f_a^2} - \frac{2\Delta f_a}{D_0 f_a} - K^{M_0} \right\} \\
& + \frac{D_b - 1}{D-2} \left\{ \frac{\langle \nabla f_b, \nabla f_b \rangle}{f_b^2} - \frac{K^{M_b}}{f_b^2} - \frac{2\Delta f_b}{D_0 f_b} - K^{M_0} \right\} \\
& + \left\{ K^{M_0} - \frac{\langle \nabla f_a, \nabla f_b \rangle}{f_a f_b} + \frac{\Delta f_a}{D_0 f_a} + \frac{\Delta f_b}{D_0 f_b} \right\}
\end{aligned}
$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \ldots$ $\quad U_1, V_1, \ldots$ $\quad U_a, V_a, \ldots$ $\quad U_n, V_n, \ldots$

$$W(X,Y,X,Y)=0 \qquad W(X,U_a,X,U_a)=0$$

$$W(U_a,U_b,U_a,U_b) \qquad W(U_a,V_a,U_a,V_a)$$

$$\begin{aligned}
\varepsilon_{U_a}\varepsilon_{U_b} W(U_a,U_b,U_a,U_b) = {} & \textstyle\sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\} \\
& + \textstyle\sum_i \sum_{j\neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\} \\
& + \textstyle\sum_{i\neq a} \frac{D_i}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_i \rangle}{f_a f_i} - \frac{\Delta f_a}{D_0 f_a} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\} \\
& + \textstyle\sum_{i\neq b} \frac{D_i}{D-2} \left\{ \frac{\langle \nabla f_b, \nabla f_i \rangle}{f_b f_i} - \frac{\Delta f_b}{D_0 f_b} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\} \\
& + \frac{D_a-1}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} - \frac{K^{M_a}}{f_a^2} - \frac{2\Delta f_a}{D_0 f_a} - K^{M_0} \right\} \\
& + \frac{D_b-1}{D-2} \left\{ \frac{\langle \nabla f_b, \nabla f_b \rangle}{f_b^2} - \frac{K^{M_b}}{f_b^2} - \frac{2\Delta f_b}{D_0 f_b} - K^{M_0} \right\} \\
& + \left\{ K^{M_0} - \frac{\langle \nabla f_a, \nabla f_b \rangle}{f_a f_b} + \frac{\Delta f_a}{D_0 f_a} + \frac{\Delta f_b}{D_0 f_b} \right\}
\end{aligned}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \ldots$ — $U_1, V_1, \ldots$ — $U_a, V_a, \ldots$ — $U_n, V_n, \ldots$

$W(X, Y, X, Y) = 0$ $\qquad$ $W(X, U_a, X, U_a) = 0$

$W(U_a, U_b, U_a, U_b) = 0$ $\qquad$ $W(U_a, V_a, U_a, V_a)$

First step: M_0 is a space of constant sectional curvature

$$M_0 \quad \times_{f_1} \quad M_1 \quad \times \quad \cdots \quad \times_{f_a} \quad M_a \quad \times \quad \cdots \quad \times_{f_n} \quad M_n$$

$X, Y, \dots$ $\quad U_1, V_1, \dots$ $\quad U_a, V_a, \dots$ $\quad U_n, V_n, \dots$

$$W(X,Y,X,Y) = 0 \qquad W(X,U_a,X,U_a) = 0$$

$$W(U_a,U_b,U_a,U_b) = 0 \qquad W(U_a,V_a,U_a,V_a)$$

$$\begin{aligned}
&\varepsilon_{U_a}\varepsilon_{V_a} W(U_a,V_a,U_a,V_a)\\
&= \sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\}\\
&\quad + \sum_i \sum_{j\neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\}\\
&\quad + \sum_{i\neq a} \frac{2D_i}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{D_0 f_a} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\}\\
&\quad + \frac{2(D_a-1)}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_a\rangle}{f_a^2} - \frac{K^{M_a}}{f_a^2} - \frac{2\Delta f_a}{D_0 f_a} - K^{M_0} \right\}\\
&\quad + \left\{ K^{M_0} - \frac{\langle \nabla f_a, \nabla f_a\rangle}{f_a^2} + \frac{K^{M_a}}{f_a^2} + \frac{2\Delta f_a}{D_0 f_a} \right\}
\end{aligned}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \quad \times_{f_1} \quad M_1 \quad \times \quad \cdots \quad \times_{f_a} \quad M_a \quad \times \quad \cdots \quad \times_{f_n} \quad M_n$$

$$X, Y, \ldots \qquad U_1, V_1, \ldots \qquad U_a, V_a, \ldots \qquad U_n, V_n, \ldots$$

$$W(X,Y,X,Y) = 0 \qquad W(X,U_a,X,U_a) = 0$$

$$W(U_a,U_b,U_a,U_b) = 0 \qquad W(U_a,V_a,U_a,V_a)$$

$$\begin{aligned}
&\varepsilon_{U_a}\varepsilon_{V_a} W(U_a,V_a,U_a,V_a) \\
&\quad = \sum_i \frac{D_i(D_i-1)}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_i \rangle}{f_i^2} + \frac{K^{M_i}}{f_i^2} + \frac{2\Delta f_i}{D_0 f_i} \right\} \\
&\qquad + \sum_i \sum_{j \neq i} \frac{D_i D_j}{(D-1)(D-2)} \left\{ K^{M_0} - \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} + \frac{\Delta f_i}{D_0 f_i} + \frac{\Delta f_j}{D_0 f_j} \right\} \\
&\qquad + \sum_{i \neq a} \frac{2D_i}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_i \rangle}{f_a f_i} - \frac{\Delta f_a}{D_0 f_a} - \frac{\Delta f_i}{D_0 f_i} - K^{M_0} \right\} \\
&\qquad + \frac{2(D_a-1)}{D-2} \left\{ \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} - \frac{K^{M_a}}{f_a^2} - \frac{2\Delta f_a}{D_0 f_a} - K^{M_0} \right\} \\
&\qquad + \left\{ K^{M_0} - \frac{\langle \nabla f_a, \nabla f_a \rangle}{f_a^2} + \frac{K^{M_a}}{f_a^2} + \frac{2\Delta f_a}{D_0 f_a} \right\}
\end{aligned}$$

First step: M_0 is a space of constant sectional curvature

$$M_0 \times_{f_1} M_1 \times \cdots \times_{f_a} M_a \times \cdots \times_{f_n} M_n$$

$X, Y, \dots$ $\quad U_1, V_1, \dots$ $\quad U_a, V_a, \dots$ $\quad U_n, V_n, \dots$

$$W(X,Y,X,Y)=0 \qquad W(X,U_a,X,U_a)=0$$

$$W(U_a,U_b,U_a,U_b)=0 \qquad W(U_a,V_a,U_a,V_a)=0$$

First step: M_0 is a space of constant sectional curvature

Theorem

$M = M_0 \times_{f_1} M_1 \times \cdots \times_{f_n} M_n$ is locally conformally flat if and only if the scaling functions satisfy

(i) $H_{f_i} = \frac{\Delta f_i}{D_0} g^{(0)}$,

(ii) $\langle \nabla f_i, \nabla f_j \rangle = f_i f_j K^{M_0} + \frac{1}{D_0} f_j \Delta f_i + \frac{1}{D_0} f_i \Delta f_j, \qquad (i \neq j)$,

(iii) $K^{M_i} = \langle \nabla f_i, \nabla f_i \rangle - \frac{2}{D_0} f_i \Delta f_i - f_i^2 K^{M_0}, \qquad (\dim M_i \geq 2)$,

where K^{M_0} and K^{M_i} denote the constant sectional curvatures of $(M_0, g^{(0)})$ and $(M_i, g^{(i)})$, respectively.

Second step: $M_0 = \mathfrak{U} \subset \mathbb{R}^{D_0}_s$ $\big(1, \overset{s}{\ldots}, 1, -1, \overset{D_0-s}{\ldots}, -1\big)$

Theorem

$M = \mathfrak{U} \times_{f_1} M_1 \times \cdots \times_{f_n} M_n$ is locally conformally flat if and only if the scaling functions satisfy

$$H_{f_i} = \frac{\Delta f_i}{D_0} g^{(0)}$$

and moreover the scaling functions are compatible in the sense that

$$\langle \nabla f_i, \nabla f_j \rangle = f_i f_j K^{M_0} + \frac{1}{D_0} f_j \Delta f_i + \frac{1}{D_0} f_i \Delta f_j \qquad (i \neq j)$$

and the sectional curvature of each internal space of dim $M_i \geq 2$ is given by

$$K^{M_i} = \langle \nabla f_i, \nabla f_i \rangle - \frac{2}{D_0} f_i \Delta f_i - f_i^2 K^{M_0}.$$

Theorem

$M = \mathfrak{U} \times_{f_1} M_1 \times \cdots \times_{f_n} M_n$ is locally conformally flat if and only if the scaling functions satisfy

$$H_{f_i} = \frac{\Delta f_i}{D_0} g^{(0)}$$

and moreover the scaling functions are compatible in the sense that

$$\langle \nabla f_i, \nabla f_j \rangle = f_i f_j K^{M_0} + \frac{1}{D_0} f_j \Delta f_i + \frac{1}{D_0} f_i \Delta f_j \qquad (i \neq j)$$

and the sectional curvature of each internal space of dim $M_i \geq 2$ is given by

$$K^{M_i} = \langle \nabla f_i, \nabla f_i \rangle - \frac{2}{D_0} f_i \Delta f_i - f_i^2 K^{M_0}.$$

Theorem

$M = \mathfrak{U} \times_{f_1} M_1 \times \cdots \times_{f_n} M_n$ is locally conformally flat if and only if the scaling functions satisfy

$$f_i(\overrightarrow{\mathbf{x}}) = a_i \langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle + \langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{x}} \rangle + c_i$$

for all $\overrightarrow{\mathbf{x}} \in \mathfrak{U}$, where $a_i, c_i \in \mathbb{R}$ and $\overrightarrow{\mathbf{b}}_i \in \mathbb{R}^{D_0}_s$, and moreover the scaling functions are compatible in the sense that

$$\langle \nabla f_i, \nabla f_j \rangle = f_i f_j K^{M_0} + \frac{1}{D_0} f_j \Delta f_i + \frac{1}{D_0} f_i \Delta f_j \qquad (i \neq j)$$

and the sectional curvature of each internal space of dim $M_i \geq 2$ is given by

$$K^{M_i} = \langle \nabla f_i, \nabla f_i \rangle - \frac{2}{D_0} f_i \Delta f_i - f_i^2 K^{M_0}.$$

Theorem

$M = \mathfrak{U} \times_{f_1} M_1 \times \cdots \times_{f_n} M_n$ is locally conformally flat if and only if the scaling functions satisfy

$$f_i(\overrightarrow{\mathbf{x}}) = a_i \langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle + \langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{x}} \rangle + c_i$$

for all $\overrightarrow{\mathbf{x}} \in \mathfrak{U}$, where $a_i, c_i \in \mathbb{R}$ and $\overrightarrow{\mathbf{b}}_i \in \mathbb{R}_s^{D_0}$, and moreover the scaling functions are compatible in the sense that

$$\langle \nabla f_i, \nabla f_j \rangle = f_i f_j K^{M_0} + \frac{1}{D_0} f_j \Delta f_i + \frac{1}{D_0} f_i \Delta f_j \qquad (i \neq j)$$

and the sectional curvature of each internal space of dim $M_i \geq 2$ is given by

$$K^{M_i} = \langle \nabla f_i, \nabla f_i \rangle - \frac{2}{D_0} f_i \Delta f_i - f_i^2 K^{M_0}.$$

Theorem

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for all $\overrightarrow{\mathbf{x}} \in \mathfrak{U}$, where $a_i, c_i \in \mathbb{R}$ and $\overrightarrow{\mathbf{b}}_i \in \mathbb{R}_s^{D_0}$, and moreover the scaling functions are compatible in the sense that

$$\langle \overrightarrow{\mathbf{b}}_i, \overrightarrow{\mathbf{b}}_j \rangle = 2(a_i c_j + a_j c_i) \qquad (i \neq j)$$

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Third step: Local structure

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Some direct consequences

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   2. *there are at most $D_0 - s + 1$ of negative curvature,*

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with scaling functions

$$\begin{aligned} f_i(\overrightarrow{\mathbf{x}}) &= x_i, \quad i = 1, \ldots, s, \\ f_{s+1}(\overrightarrow{\mathbf{x}}) &= 1 - \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle, \\ f_i(\overrightarrow{\mathbf{x}}) &= x_{i-1}, \quad i = s+2, \ldots, D_0 + 1, \\ f_{D_0+2}(\overrightarrow{\mathbf{x}}) &= 1 + \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle. \end{aligned}$$

is a locally conformally flat MCM.

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\end{aligned}$$

is a locally conformally flat MCM.

Example with compact internal space as small as desired

Let $\mathfrak{U}$ be a suitable open subset in $\mathbb{R}_s^{D_0}$. Then,

$$M = \mathfrak{U} \times_{f_1} \mathbb{F}_1^{D_1} \times \cdots \times_{f_{s+1}} \mathbb{F}_{s+1}^{D_{s+1}} \times_{f_{s+2}} \mathbb{F}_{s+2}^{D_{s+2}} \times \cdots \times_{f_{D_0+2}} \mathbb{F}_{D_0+2}^{D_{D_0+2}}$$

with scaling functions

$$\begin{aligned} f_i(\overrightarrow{\mathbf{x}}) &= x_i, \quad i = 1, \ldots, s, \\ f_{s+1}(\overrightarrow{\mathbf{x}}) &= 1 - \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle, \\ f_i(\overrightarrow{\mathbf{x}}) &= x_{i-1}, \quad i = s+2, \ldots, D_0 + 1, \\ f_{D_0+2}(\overrightarrow{\mathbf{x}}) &= 1 + \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle. \end{aligned}$$

is a locally conformally flat MCM.

Example with compact internal space as small as desired

Let $\mathfrak{U}$ be a suitable open subset in $\mathbb{R}^{D_0}_s$. Then,

$$M = \mathfrak{U} \times_{f_1} \mathbb{F}^{D_1}_{1} \times \cdots \times_{f_{s+1}} \mathbb{F}^{D_{s+1}}_{s+1} \times_{f_{s+2}} \mathbb{F}^{D_{s+2}}_{s+2} \times \cdots \times_{f_{D_0+2}} \mathbb{F}^{D_{D_0+2}}_{D_0+2}$$

with scaling functions

$$\begin{aligned} f_i(\overrightarrow{\mathbf{x}}) &= \alpha_i \; x_i, \quad i = 1, \ldots, s, \\ f_{s+1}(\overrightarrow{\mathbf{x}}) &= \alpha_{s+1} \; (1 - \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle), \\ f_i(\overrightarrow{\mathbf{x}}) &= \alpha_i \; x_{i-1}, \quad i = s+2, \ldots, D_0 + 1, \\ f_{D_0+2}(\overrightarrow{\mathbf{x}}) &= \alpha_{D_0+2} \; (1 + \tfrac{1}{4}\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{x}} \rangle). \end{aligned}$$

is a locally conformally flat MCM.

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