

MODULAR CLASS OF EVEN SYMPLECTIC MANIFOLDS

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We provide an intrinsic description of the notion of modular class for an even symplectic manifold and study its properties in this coordinate-free formalism.

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1. Introduction

The definition of the modular vector field of a Poisson manifold $(M; \{_, _ \})$ is as follows: given a volume element η on M , the modular vector field Z^M maps each function $f \in C^\infty(M)$ to the divergence with respect to η of the Hamiltonian vector field associated with f , i.e.,

$$Z^M(f) := \operatorname{div}^\eta(X_f) = \operatorname{div}^\eta(\{df, _ \}). \quad (1)$$

The modular class of $(M; \{_, _ \})$ is its class in the Poisson–Lichnerowicz cohomology [1]. Koszul [2] introduced the concept of modular vector field in his study of the cohomology of a Poisson manifold, and Weinstein [3] used it to understand the modular automorphisms of von Neumann algebras, observing that these and their semiclassical limits (Poisson algebras) share the property of having modular automorphism groups. The concept also appeared in geometry in the classification of quadratic Poisson structures (see [4]). The modular vector field and the related notion of volume element were also intensively used by O. M. Khudavardian and others to study graded Poincaré–Cartan invariants, the geometry of the Batalin–Vilkovisky formalism, etc. (see [5]–[7]). We therefore feel that an intrinsic, geometric study of these structures deserves attention.

The notion of the modular class only needs a Poisson structure to be defined, but we focus our attention on the nondegenerate case. In the graded formalism, when a graded Poisson manifold $((M, \wedge\mathcal{E}), \llbracket _, _ \rrbracket)$ is given (see Secs. 2 and 3 for the definitions), a fundamental distinction appears: even though an appropriate definition of the divergence can be given, the analogue of mapping (1) when the Poisson bracket is odd with respect to the $\mathbb{Z}$ -grading gives not a derivation on $\wedge\mathcal{E}$ but a generator for the Poisson bracket in the sense of Gerstenhaber algebras (see [6], [8], or [9]). On the other hand, when the bracket is even with respect to the $\mathbb{Z}$ -grading, the same mapping does give a derivation on $\wedge\mathcal{E}$. It therefore makes sense to develop the notions of graded modular vector field and modular class in precisely this case.

In the nongraded case, it is well known that any symplectic manifold (M, ω) is unimodular, i.e., it gives the zero class. We now suppose that M is the base manifold of a given graded Poisson manifold $(M, \wedge\mathcal{E})$ whose graded Poisson bracket $\llbracket _, _ \rrbracket$ is nondegenerate and extends the Poisson bracket on M defined by ω . It is also known that this bracket has an associated volume form that is expressed in local coordinates by the Berezinian of the bracket matrix (see [10]); using this volume form, we can see that the modular vector field is zero and $(M, \wedge\mathcal{E})$ is therefore unimodular.

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Our purpose in this paper is to give a geometric, coordinate-free formalism for these results. We define the notion of a symplectic Berezinian volume element intrinsically and study how it changes with the choice of the Berezinian sheaf section, along with its relation to the canonical Berezinian. As an application, we give a graded formulation of the continuity equation in fluid mechanics.

2. Graded forms on $(M, \Gamma(\wedge E))$

Generalities concerning graded manifolds can be found in [10]–[12]; our approach follows [13]. Let M be an m -dimensional smooth manifold, and let C_M^∞ be the sheaf of smooth functions on M . Let $E \rightarrow M$ be a vector bundle of rank n , and let $\mathcal{E} = \Gamma(E)$ be its sheaf of smooth sections. Let $\wedge \mathcal{E} = \Gamma(\wedge E)$ be the sheaf of smooth sections of the exterior algebra bundle $\wedge E \rightarrow M$. We refer to [11] or [13] for the definitions of a graded vector field, a graded differential form, the insertion operator $\iota(D)$ (D being a graded vector field), the exterior differential d^G , and the Lie operator $\mathcal{L}_D^G$.

Being a graded homomorphism of graded modules, a graded differential form has a degree. We can therefore define a $\mathbb{Z} \times \mathbb{Z}$ bigrading on the module of graded differential forms, and we say that a graded differential form λ has the bidegree $(p, k) \in \mathbb{Z} \times \mathbb{Z}$ if

$$\lambda: \underbrace{\text{Der } \wedge \mathcal{E} \times \cdots \times \text{Der } \wedge \mathcal{E}}_{p \text{ times}} \longrightarrow \wedge \mathcal{E}$$

and if

$$|\langle D_1, \dots, D_p; \lambda \rangle| = \sum_{i=1}^p |D_i| + k$$

for all $D_1, \dots, D_p \in \text{Der } \wedge \mathcal{E}$. Using this bigrading, we can decompose any graded p -differential form λ into a sum $\lambda = \lambda_{(0)} + \cdots + \lambda_{(n)}$, where $\lambda_{(i)}$ is a homogeneous graded form of the bidegree (p, i) .

The fundamental result follows from a theorem by Kostant [11].

Proposition 1. *Every d^G -closed graded form of the bidegree (p, k) with $k > 0$ is exact.*

We use the fact that the space $\text{Der } \wedge \mathcal{E}$ of graded vector fields is a locally free sheaf of $\wedge \mathcal{E}$ -modules [11] (see [14] and [15] for an analysis of its structure). Let $\mathcal{E}^*$ be the sheaf of sections of the dual bundle $E^* \rightarrow M$. There is a monomorphism

$$i: \Gamma(\wedge \mathcal{E}) \otimes \mathcal{E}^* \hookrightarrow \text{Der } \wedge \mathcal{E}.$$

On the other hand, let $\mathcal{X}(M) = \text{Der } C_M^\infty$ be the sheaf of smooth vector fields on M . A connection ∇ on $\wedge \mathcal{E}$, by definition, gives a morphism

$$\Gamma(\wedge \mathcal{E}) \otimes \mathcal{X}(M) \rightarrow \text{Der } \wedge \mathcal{E}, \quad \alpha \otimes X \mapsto \alpha \nabla_X.$$

3. Divergence operators and modular graded vector fields

By definition, a divergence operator on $\wedge \mathcal{E}$ is an even linear map $\text{div}: \text{Der } \wedge \mathcal{E} \rightarrow \wedge \mathcal{E}$ such that

$$\text{div}(sD) = s \text{div}(D) + (-1)^{|s|} |D| D(s) \quad (2)$$

for any $D \in \text{Der } \wedge \mathcal{E}$ and any $s \in \wedge \mathcal{E}$. The modular vector field Z^M associated with a divergence operator div and a graded Poisson bracket $\llbracket -, - \rrbracket$ on $\wedge \mathcal{E}$ is the even graded vector field defined as

$$s \in \wedge \mathcal{E} \mapsto D_s = \llbracket s, - \rrbracket \in \text{Der } \wedge \mathcal{E} \mapsto \text{div}(D_s) \in \wedge \mathcal{E}. \quad (3)$$

It is easy to verify that when the even Poisson bracket is the Poisson bracket associated with an even symplectic form Θ , then Z^M is a locally Hamiltonian graded vector field. From now on, we work exclusively with this case, i.e., with an even symplectic form on $(M, \wedge \mathcal{E})$ and its associated even Poisson bracket $\llbracket -, - \rrbracket_\Theta$.

Lemma 1. *Let $D = \sum_{i \in \mathbb{N}} D_{2i} \in \text{Der } \wedge \mathcal{E}$ be a locally Hamiltonian even derivation. We consider the decomposition (with respect to the $\mathbb{Z}$ -degree) $\Theta = \Theta_{(0)} + \Theta_{(\geq 2)}$. Then D is a graded Hamiltonian vector field for Θ if and only if $\iota_{D_0} \Theta_{(0)}$ is an exact graded form.*

Proof. The proof is an straightforward computation using Proposition 1.

It follows from Lemma 1 that the modular class depends on only the zeroth-degree term of the modular vector field.

4. The symplectic Berezinian volume element and the modular class

Let Θ be an even symplectic form on a graded manifold $(M, \wedge \mathcal{E})$ of dimension $(2n, m)$. We know that there are three objects associated with the even symplectic form (see [15]): the usual symplectic form ω on the base manifold M , the nondegenerate symmetric bilinear form g on E^* , and the connection ∇ on E compatible with g , i.e., $\nabla g = 0$.

Let ω^n be the symplectic volume element on M , and let μ_g be the metric volume element on E . Given $s \in \wedge \mathcal{E}$ of compact support, we can define

$$\int_{\xi} s := \int_M (i_{\mu_g} s) \omega^n,$$

where $i_{\mu_g} s$ denotes the total contraction of $\mu_g \in \Gamma(\Lambda^m E^*)$ with s . Such a definition implicitly includes the definition of a Berezinian volume element ξ [9], [12], [16].

We define a divergence operator associated with the even symplectic form through the Berezinian volume element ξ . Given a derivation $D \in \text{Der } \wedge \mathcal{E}$, there is a unique section, denoted by $\text{div}^{\xi}(D) \in \wedge \mathcal{E}$, such that

$$-\int_{\xi} D(s) = \int_{\xi} \text{div}^{\xi}(D) \wedge s$$

for all $s \in \wedge \mathcal{E}$ of compact support. This is indeed a divergence operator.

Proposition 2. *The formula*

$$\text{div}^{\xi}(s \wedge D) = s \wedge \text{div}^{\xi}(D) + (-1)^{|D||s|} D$$

holds.

Proof. The proof is just a matter of computation:

$$\begin{aligned} \int_{\xi} \text{div}^{\xi}(s \wedge D) \wedge \bar{s} &= - \int_{\xi} s \wedge D(\bar{s}) = - \int_M i_{\mu_g}(s \wedge D(\bar{s})) \omega^n = \\ &= - (-1)^{|D||s|} \int_M i_{\mu_g}(D(s \wedge \bar{s})) \omega^n + \\ &\quad + (-1)^{|D||s|} \int_M i_{\mu_g}(D(s) \wedge \bar{s}) \omega^n = \\ &= - (-1)^{|D||s|} \int_{\xi} D(s \wedge \bar{s}) + (-1)^{|D||s|} \int_{\xi} D(s) \wedge \bar{s} = \\ &= (-1)^{|D||s|} \int_{\xi} \text{div}^{\xi}(D) \wedge s \wedge \bar{s} + (-1)^{|D||s|} \int_{\xi} D(s) \wedge \bar{s} = \\ &= \int_{\xi} (s \wedge \text{div}^{\xi}(D) + (-1)^{|D||s|} D(s)) \wedge \bar{s}. \end{aligned}$$

We now want to know what happens when we change the section of the Berezinian sheaf; for this, we recall that the Berezinian module is a right $\wedge\mathcal{E}$ -module of rank 1 (see [16]). Therefore, given a Berezinian volume element ξ , any other Berezinian volume element has the form $\xi\bar{s}$ for an invertible even element, $\bar{s} \in \wedge\mathcal{E}$.

Proposition 3. *If $\bar{s}$ is of compact support, then $\operatorname{div}^{\xi\bar{s}} = \operatorname{div}^{\xi} + d^G \log \bar{s}$.*

Proof. From the definition of the Berezinian, we have

$$\int_{\xi\bar{s}} \cdot = \int_{\xi} \bar{s} \wedge \cdot.$$

We now have

$$\int_{\xi\bar{s}} D(s) = - \int_{\xi\bar{s}} \operatorname{div}^{\xi\bar{s}}(D) \wedge s = - \int_{\xi} \bar{s} \wedge \operatorname{div}^{\xi\bar{s}}(D) \wedge s \quad (4)$$

for any $s \in \wedge\mathcal{E}$. On the other hand,

$$\begin{aligned} \int_{\xi\bar{s}} D(s) &= \int_{\xi} \bar{s} \wedge D(s) = \int_M i_{\mu_g}(\bar{s} \wedge D(s)) \omega^n = \\ &= \int_M i_{\mu_g}(D(\bar{s} \wedge s)) \omega^n - \int_M i_{\mu_g}(D(\bar{s}) \wedge s) \omega^n = \\ &= \int_{\xi} D(\bar{s} \wedge s) - \int_{\xi} D(\bar{s}) \wedge s = \\ &= - \int_{\xi} \operatorname{div}^{\xi}(D) \wedge \bar{s} \wedge s - \int_{\xi} \bar{s} \wedge \bar{s}^{-1} \wedge D(\bar{s}) \wedge s = \\ &= - \int_{\xi} \bar{s} \wedge \operatorname{div}^{\xi}(D) \wedge s - \int_{\xi} \bar{s} \wedge \bar{s}^{-1} \wedge D(\bar{s}) \wedge s. \end{aligned} \quad (5)$$

Equating (4) and (5), we obtain

$$\begin{aligned} \operatorname{div}^{\xi\bar{s}}(D) &= \operatorname{div}^{\xi}(D) + \bar{s}^{-1} \wedge D(\bar{s}) = \\ &= \operatorname{div}^{\xi}(D) + D(\log \bar{s}) = \\ &= \operatorname{div}^{\xi}(D) + \langle D; d^G \log \bar{s} \rangle \end{aligned}$$

and hence the statement.

The proved Proposition 3 allows giving the following definition of a modular class.

Definition. The modular class of an even Poisson bracket is the class of any modular vector field in the quotient $\operatorname{Der} \wedge\mathcal{E} / \operatorname{Ham}(\pi)$.

We note how the notion of the symplectic Berezinian is related to that of the canonical Berezinian. For the given volume form ω^n on M and the metric volume μ_g , because they are forms of the maximal degree on M , there must exist a function f such that $\omega^n = e^f \mu_g$. If $s_{(\max)}$ denotes the maximal-degree part of the section s , then there must also exist an h with $s_{(\max)} = h \mu_g$, and the canonical Berezinian gives

$$\int_{\operatorname{can}} s = \int_M s_{(\max)}.$$

On the other hand, the symplectic Berezinian is

$$\begin{aligned} \int_{\text{symp}} s &= \int_M (i_{\mu_g} s)_{(\max)} \omega^n = \int_M i_{\mu_g} (h\mu_g) e^f \mu_g = \\ &= \int_M h e^f \mu_g = \int_M e^f s_{(\max)}, \end{aligned}$$

and e^f is therefore the section that passes from $\int_{\text{symp}}$ to $\int_{\text{can}}$. By Proposition 3, the associated divergences are then related through

$$\text{div}^{\text{symp}} = \text{div}^{\text{can}} + d^G f.$$

In the case where (M, ω, g) is a Kähler manifold, in which f is a constant function, $\text{div}^{\text{symp}} = \text{div}^{\text{can}}$.

The basic derivations in this setting are of the form i_χ for $\chi \in \Gamma(E^*)$ and ∇_X for a vector field X , where we can use the linear connection ∇ induced by the even symplectic form. We compute their divergences.

Lemma 2. *If ∇ is a connection compatible with g , then*

$$\text{div}^\xi(i_\chi) = 0, \quad \text{div}^\xi(\nabla_X) = \text{div}^{\omega^n}(X).$$

Proof. Indeed, $i_\chi s$ is a section of degree less than $m = \text{rk}(E)$; hence, $i_{\mu_g} i_\chi s = 0$ for any s . For the other basic derivations, we have

$$\begin{aligned} i_{\mu_g}(\nabla_X s) \omega^n &= X(i_{\mu_g} s) \omega^n - (i_{\nabla_X \mu_g} s) \omega^n = \\ &= \mathcal{L}_X((i_{\mu_g} s) \omega^n) - (i_{\mu_g} s) \mathcal{L}_X \omega^n = \\ &= d i_X((i_{\mu_g} s) \omega^n) + i_X d((i_{\mu_g} s) \omega^n) - (i_{\mu_g} s) \mathcal{L}_X \omega^n. \end{aligned}$$

Now, the first term, $d i_X((i_{\mu_g} s) \omega^n)$, does not contribute to the integral because it is an exact term. The second term, $i_X d((i_{\mu_g} s) \omega^n)$, is equal to zero because $(i_{\mu_g} s) \omega^n$ is a top-degree differential form on M . The third term gives $(i_{\mu_g} s) \text{div}^{\omega^n}(X) \omega^n$ because $\mathcal{L}_X \omega^n = \text{div}^{\omega^n}(X) \omega^n$. Finally, we note that $\nabla_X \mu_g$ vanishes by hypothesis. Therefore,

$$- \int_\xi \nabla_X s = \int_\xi \text{div}^{\omega^n}(X) s.$$

The lemma is proved.

Theorem. *Any even symplectic form on a graded manifold $(M, \wedge \mathcal{E})$ is unimodular.*

Proof. We recall the Rothstein theorem: $\Theta = \varphi^*(\Theta_{\omega, g, \nabla})$ for an automorphism φ of $\wedge \mathcal{E}$, where ∇ is compatible with g ; it is therefore clear that if we prove that $\Theta_{\omega, g, \nabla}$ is unimodular, then Θ is also unimodular. Now, $\Theta_{\omega, g, \nabla}$ is given by

$$\begin{aligned} \langle \nabla_X, \nabla_Y; \Theta_{\omega, g, \nabla} \rangle &= \omega(X, Y) + \frac{1}{2} R(X, Y, -, -), \\ \langle \nabla_X, i_\chi; \Theta_{\omega, g, \nabla} \rangle &= 0, \\ \langle i_\chi, i_\psi; \Theta_{\omega, g, \nabla} \rangle &= g(\chi, \psi). \end{aligned}$$

Therefore, $(\Theta_{\omega, g, \nabla})_{(0)}$, denoted by $\Theta_{(0)}^{\omega, g, \nabla}$, is given by

$$\begin{aligned} \langle \nabla_X, \nabla_Y; \Theta_{(0)}^{\omega, g, \nabla} \rangle &= \omega(X, Y), \\ \langle \nabla_X, i_\chi; \Theta_{(0)}^{\omega, g, \nabla} \rangle &= 0 = \langle i_\chi, i_\psi; \Theta_{(0)}^{\omega, g, \nabla} \rangle. \end{aligned}$$

The graded Hamiltonian vector field associated with $f \in C^\infty(M)$ through the symplectic form $\Theta_{\omega,g,\nabla}$ is given by

$$D_f = \nabla_{X_f} + \text{h.d.t.}$$

It follows from Lemma 1 that D is a graded Hamiltonian vector field if and only if $\iota_{D_0} \Theta_{(0)}$ is an exact graded form. On the other hand, we know that

$$\pi_{(0)}(Z^M(f)) = \pi_{(0)}(\text{div}(D_f)) = \text{div}(\nabla_{X_f}) = 0. \quad (6)$$

Therefore, $Z^M = i_N + \text{h.d.t.}$, where $N \in \text{End } \mathcal{E}$. But then

$$\iota_{Z_0^M} \Theta_{(0)}^{\omega,g,\nabla} = \iota_{i_N} \Theta_{(0)}^{\omega,g,\nabla} = 0.$$

The theorem is proved.

5. Applications

In this section, we intend to provide some ideas about the possible applications of these results. In the classical case, the notions of divergence and vanishing modular class are intimately related to conservation laws along the flow of fluids, in fact, to one of the basic equations of fluid dynamics, the continuity equation. We do not intend to give a complete description of the equations of graded fluids here; we content ourselves with a study of what the graded continuity equation must be (this is the only basic equation of fluid dynamics that is directly related to the conservation of volume by the Hamiltonian flow).

We consider more specifically the classical situation we want to extend to the graded case. Let $V \in \mathcal{X}(M)$ be a vector field describing a classical dynamic system (for instance, the velocity field on a fluid), and let $\{\varphi_t\}_{t \in \mathbb{R}}$ be its flow. Associated with any function $f \in C^\infty(M)$ (which describes the density of some observable in the system), we have the continuity equation

$$\frac{\partial f}{\partial t} + \text{div}(fV) = 0, \quad (7)$$

where we allow the possibility that f is time dependent. This equation expresses the conservation of the total magnitude associated with f ,

$$\frac{d}{dt} \int_M f \mu = 0, \quad (8)$$

where μ is a volume form on M , usually the symplectic volume form coming from the Hamiltonian structure of the dynamic system.

What would be the graded analogue of (7)? We cannot repeat the physical reasoning of the classical case, because in the graded case, there is no notion of volume form (understood as a maximal-degree graded form), but we can extend the geometric interpretation. For this, we note that (7) can be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_V \right) (f\mu) = 0. \quad (9)$$

The continuity equation in form (9) allows interpreting $f\mu$ as a density form on the fluid that is dynamically conserved along the flow $\{\varphi_t\}_{t \in \mathbb{R}}$. Here f can be a volume density, a charge density, etc. Moreover, this equation and its geometric interpretation carry over to graded manifolds. Now, an “observable density” is a superfunction $\rho \in \wedge \mathcal{E}$. A graded vector field is $D \in \text{Der } \wedge \mathcal{E}$, and its flow, in general, is a two-parameter

dependent $\{\Phi_{(t,s)}^*\}_{(t,s) \in \mathbb{R}^{1|1}}$, where $\Phi: \mathbb{R}^{1|1} \times (M, \wedge \mathcal{E}) \rightarrow (M, \wedge \mathcal{E})$ (see [17] for details on superflows). Thus, if (t, s) are the (global) supercoordinates on $\mathbb{R}^{1|1}$, the graded analogue of (9) would be the expression of the conservation of ρ along the flow of D :

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \mathcal{L}_D^G \right) (\rho) = 0, \quad (10)$$

where we take $\partial/\partial t + \partial/\partial s$ as the “integrating model” for supervector field flows (see [17]). Also, ρ can eventually depend on s and t .

Using our results (Proposition 2 and the theorem), we can recast (10) in a form similar to classical form (7):

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \mathcal{L}_D^G \right) (\rho) &= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) (\rho) + D(\rho) = \\ &= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) (\rho) + (-1)^{|D||\rho|} (\operatorname{div}^\xi(\rho D) - \rho \wedge \operatorname{div}^\xi(D)) = \\ &= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) (\rho) + (-1)^{|D||\rho|} (\operatorname{div}^\xi(\rho D)). \end{aligned}$$

The equation of continuity therefore becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) (\rho) + (-1)^{|D||\rho|} (\operatorname{div}^\xi(\rho D)) = 0.$$

Indeed, although it is not obvious, this equation is of the “conservation of mass” type. We need only take the properties of the superflows, which are analogues of those of the classical flow of vector fields, into account. We let $(U, \wedge \mathcal{E}|_U)$ denote an open superdomain and $\Phi_{(t,s)}^*(U, \wedge \mathcal{E}|_U)$ denote the superdomain obtained from the action of the superflow of D . Then if $\int$ denotes the Berezinian integral, then we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \int_{\Phi_{(t,s)}^*(U, \wedge \mathcal{E}|_U)} \rho &= \int_{(U, \wedge \mathcal{E}|_U)} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \Phi_{(t,s)}^* \rho = \\ &= \int_{(U, \wedge \mathcal{E}|_U)} \Phi_{(t,s)}^* \left[\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \rho + \mathcal{L}_D^G \rho \right] = \\ &= \int_{\Phi_{(t,s)}^*(U, \wedge \mathcal{E}|_U)} \left[\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \rho + \mathcal{L}_D^G \rho \right], \end{aligned}$$

and the continuity equation is equivalent to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \int \rho = 0, \quad (11)$$

which is a conservation equation.

We note how this result embodies the classical result for the conservation of mass in a fluid (for definiteness, moving on $\mathbb{R}^2$ with its usual symplectic and metric structure). It suffices to take $\rho(\vec{x}, t) = f(\vec{x}, t)\mu$ (where f is the density of the fluid and μ is the symplectic volume form on $\mathbb{R}^2$) and $D = \mathcal{L}_X$ (where X is the field of velocities) as a derivation on $(\mathbb{R}^2, \Gamma(\wedge T^*\mathbb{R}^2))$. By the definition of the Berezinian integral, Eq. (11) then leads to

$$0 = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \rho + \mathcal{L}_D^G \rho = \frac{\partial}{\partial t} (f\mu) + \mathcal{L}_X (f\mu) = \frac{\partial f}{\partial t} \mu + \operatorname{div}(fX)\mu,$$

i.e., to the classical equation

$$\frac{\partial f}{\partial t} + \operatorname{div}(fX) = 0.$$

The advantage of Eq. (11) is that it allows considering all kinds of magnitudes expressible as differential forms, in the spirit of the generalization of classical mechanics proposed by Michor (see [18]).

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