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Composition operators on weighted spaces of holomorphic functions on JB^*-triples

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Abstract We characterise continuity of composition operators on weighted spaces of holomorphic functions $H_v(B_X)$, where B_X is the open unit ball of a Banach space which is homogeneous, that is, a JB^*-triple.

Keywords Yang-Baxter equation · JB*-triples · composition operator

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1 Introduction

In this note, we prove a result concerning composition operators on JB^*-triples. These triples are Banach spaces which carry a certain algebraic structure.

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They form quite a large class, including Hilbert spaces and C^*-algebras (see example 1 below), and are interesting from both the mathematical and the physical point of view. On the mathematical side, they play a rôle similar to that of semisimple Lie algebras in the study of symmetric finite dimensional manifolds, but in the context of infinite dimensional spaces (see [26] and references therein). Also, JB^*-triples are intimately related to Jordan algebras, which are long known to appear in quantum mechanics (see [12, 20, 16], or [27] for a recent account). JB^*-triples have been found to be useful in solving Yang-Baxter equations ([22]), constructing Lie superalgebras (see [17] and [24]) and in the study of multifield integrable systems (see [1] or [25] and references therein).

With respect to composition operators, let us recall that on a classical level the coherent states of a physical system are described by holomorphic functions on the classical phase space (see [4]). When passing to the quantum framework, one deals with the general concept of state over $\mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$), which is a normalized positive linear functional on $\mathcal{B}(\mathcal{H})$ (see [2]). In these contexts, composition operators can be seen as "dictionaries" translating these states from one reference frame to another when we have a holomorphic transformation between the underlying spaces $\phi : X \to Y$ (in this case, the composition operator associated to ϕ, C_ϕ, is a map $C_\phi : H(Y) \to H(X)$, where $H(X)$ is the space of holomorphic mappings from X to $\mathbb{C}$).

Both situations (the classical and the quantum ones), are generalized in the study of weighted spaces of holomorphic functions on the unit ball B of a Banach space X, denoted $H_v(B)$. These spaces have been widely studied in recent years, and are quite well understood. The first case considered was that of B being the unit disc or a domain in $\mathbb{C}$ or $\mathbb{C}^n$. Special interest has been given to the study of composition operators between these spaces; we refer to [6, 8, 9] and particularly to the recent surveys [5, 7] and the references therein for information about the subject. Some study has also been devoted to the situation when B_X is the open unit ball of a Banach space X (see e.g. [3, 13, 14]). Some of the results in [8] were generalised in [14] to the Banach space setting. One result given in [14] characterizes continuity of composition operators when B is the open unit ball of a Hilbert space. The proof relies on the fact that there exist enough automorphisms of B. In this note, we show that this requirement is also fulfilled if we consider unit balls of JB^*-triples.

2 Preliminary results

We begin by fixing notation and some results; for details see [14]. Let X be a Banach space and B_X its open unit ball. By a weight we mean any continuous bounded mapping $v : B_X \to]0, \infty[$. We denote by $H(B_X)$ the space of holomorphic functions $f : B_X \longrightarrow \mathbb{C}$. A set $A \subset B_X$ is said to be B_X-bounded if $d(A, X \setminus B_X) > 0$. The subspace of $H(B_X)$ consisting of those functions which are bounded on the B_X-bounded sets is denoted by $H_b(B_X)$.

Following [8] and [13] we consider

$$H_v(B_X) = \{f \in H(B_X) : \|f\|_v = \sup_{x \in B_X} v(x)|f(x)| < \infty\},$$

where v is a weight. With the norm $\| \ \|_v$, the space $H_v(B_X)$ is a Banach space.

Given a weight v, we consider the following associated weight $\tilde{v}(x) = 1/\sup_{\|f\|_v \leq 1} |f(x)|$ (see [6,8,14]). We say that a weight v is norm-radial if $v(x) = v(y)$ for every x, y such that $\|x\| = \|y\|$. If v is norm-radial and non-increasing (with respect to the norm) then $\tilde{v}$ is also norm-radial and non-increasing.

A weight v satisfies Condition I if $\inf_{x \in rB_X} v(x) > 0$ for every $0 < r < 1$ ([13]). If v satisfies Condition I, then $H_v(B_X) \subseteq H_b(B_X)$ ([13, Proposition 2]).

Definition 1 Let X and Y be Banach spaces and $\phi : B_X \to B_Y$ a holomorphic mapping. The composition operator associated to ϕ is defined by

$$C_\phi : H(B_Y) \longrightarrow H(B_X) \quad , \quad f \rightsquigarrow C_\phi(f) = f \circ \phi.$$

C_ϕ is clearly linear. Denoting by τ_0 the compact-open topology, C_ϕ is also (τ_0, τ_0)-continuous. Given any two weights v_X, v_Y defined on B_X, B_Y respectively, we consider the restriction $C_\phi : H_{v_Y}(B_Y) \to H_{v_X}(B_X)$ whenever this is well defined. It is known that if C_ϕ is well defined, then it is continuous (see [14]). The following result was proved in [14] (see also [8, Proposition 2.1]).

Proposition 1 *Let v_X, v_Y be two weights satisfying Condition I and $\phi : B_X \longrightarrow B_Y$ holomorphic. Then the following are equivalent,*
(i) $C_\phi : H_{v_Y}(B_Y) \longrightarrow H_{v_X}(B_X)$ *is well defined and continuous.*
(ii) $\displaystyle\sup_{x \in B_X} \frac{v_X(x)}{\tilde{v}_Y(\phi(x))} < \infty.$
(iii) $\displaystyle\sup_{x \in B_X} \frac{\tilde{v}_X(x)}{\tilde{v}_Y(\phi(x))} < \infty.$
(iv) $\displaystyle\sup_{\|\phi(x)\| > r_0} \frac{v_X(x)}{\tilde{v}_Y(\phi(x))} < \infty$ *for some* $0 < r_0 < 1$.

3 JB^*-triples

We intend to study composition operators on a JB^*-triple X. In this case, B_X is a bounded symmetric domain. Given a domain D in a Banach space, a symmetry at $a \in D$ is a biholomorphic map $s_a : D \to D$ such that $s_a^2 = id$ and $s_a(a) = a$ is an isolated fixed point. A bounded symmetric domain is a bounded domain (or a domain biholomorphically equivalent to a bounded domain) which has a symmetry at every point.

Definition 2 A JB^*-triple is a Banach space Z with a triple product $\{\ ,\ ,\ \} : Z^3 \longrightarrow Z$ that is linear and symmetric in the first and third variables (symmetric in the sense that $\{x, y, z\} = \{z, y, x\}$ for all x, z) and antilinear in the second variable and which satisfies,
(i) the mapping $x\Box x$, given by $x\Box x(z) = \{x, x, z\}$ is Hermitian, $\sigma(x\Box x) \geq 0$ and $\|x\Box x\| = \|x\|^2$,
(ii) for every $a, b, x, y, z \in X$, the Jordan triple identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

holds.

For $x, y \in Z$, we define three mappings $x\Box y$ (linear), Q_x (antilinear) and $B(x, y)$ (linear) by

$$x\Box y(z) = \{x, y, z\},$$
$$Q_x(z) = \{x, z, x\},$$
$$B(x, y) = id - 2x\Box y + Q_xQ_y.$$

We also consider the operator $B_x = B(x, x)^{1/2}$ (the square root taken in the sense of functional calculus, i.e. $B_x \circ B_x = B(x, x)$). It is known that ([19])

$$\|B_x^{-1}\| = \frac{1}{1 - \|x\|^2}. \tag{1}$$

For background on JB^*-triples, see [15, 21].

It is a well known fact that the open unit ball of a Banach space is symmetric if and only if the space is a JB^*-triple [18]. Also, a bounded domain D is symmetric if and only if it has a transitive group of biholomorphic mappings $\{g_a\}_{a\in D}$ and a symmetry at some point p. In this case the bounded symmetric domain is biholomorphically equivalent to the unit ball of a JB^*-triple and all biholomorphic mappings on the unit ball can be explicitly described. They are of the form Kg_a where K is a surjective linear isometry and g_a are Möbius type mappings that satisfy $g_a(0) = a$ and $g_a^{-1} = g_{-a}$ ([19]). These mappings can be defined from the triple product by

$$\begin{aligned} g_a(x) &= a + (B(a, a)^{1/2} \circ B(x, a)^{-1})(x - Q_x(a)) \\ &= a + B_a(\sum_{n=0}^{\infty}(-x\Box a)^n a) \end{aligned}$$

If s_0 denotes the symmetry at 0 (i.e. $x \mapsto -x$), the symmetry at any other point of the unit ball a is given by $g_a \circ s_0 \circ g_{-a}$.

Example 1 Examples of JB^*-triples are Hilbert spaces and C^*-algebras. On a Hilbert space the triple product is given by $\{x, y, z\} = 1/2((x|y)z + (z|y)x)$. The Möbius mappings for Hilbert spaces were defined by Renaud in [23]. If Z is a C^*-algebra, the triple product is given by $\{x, y, z\} =$

$1/2(xy^*z + zy^*x)$. Another example of JB^*-triples that includes the two previous ones are J^*-algebras, that is closed subspaces of $\mathcal{L}(H, K)$ (H and K Hilbert spaces) which are closed under $A \mapsto AA^*A$ (cf. [15]).

As already mentioned, the symmetries of a bounded symmetric domain can be defined using a set of Möbius-like mappings. Let us show that these vector Möbius mappings behave in the same way as the scalar ones when we take the supremum on a sphere (a circle in the scalar case).

Lemma 1 *Let B be a bounded symmetric domain (i.e., the open unit ball of a JB^*-triple Z) and $\{g_a\}_{a\in B}$ the transitive group of biholomorphic mappings that define the symmetries. Then, for each $0 < r < 1$*

$$\sup_{\|x\|=r} \|g_a(x)\| = \frac{\|a\| + r}{1 + r\|a\|}$$

and this supremum is attained at some point.

Proof First, for any bounded symmetric domain we show that $\|g_a(x)\| \le \frac{\|a\|+\|x\|}{1+\|a\|\cdot\|x\|}$. It is well known ([21]) that

$$\frac{1}{1 - \|g_a(x)\|^2} = \|B_a^{-1} \circ B(a, x) \circ B_x^{-1}\|.$$

In particular, using (1) we get

$$\frac{1}{1 - \|g_a(x)\|^2} \le \frac{1}{1 - \|a\|^2}\ (1 + \|a\| \cdot \|x\|)^2\ \frac{1}{1 - \|x\|^2}.$$

Hence

$$\|g_a(x)\| \le \frac{\|a\| + \|x\|}{1 + \|a\| \cdot \|x\|}.$$

Next we show that the bound is attained, in the sense that there exists $x \in B$, $\|x\| = r$ with $\|g_a(x)\| = \frac{\|a\|+r}{1+r\ \|a\|}$. Clearly we may assume $a \neq 0$. Let us consider Z_a the JB^*-subtriple of Z generated by a, that is, the smallest (closed) JB^*-subtriple of Z that contains a. It is obviously enough to find $x \in Z_a$ attaining the bound. A result of Kaup ([18, Proposition 5.3]) shows that for any JB^*-triple and $a \in Z$, Z_a is isometrically (triple) isomorphic to $C_0(\Omega)$, where $\Omega \subseteq \mathbb{R}^+$ satisfies $\Omega \cup \{0\}$ is compact. The Möbius maps on the unit ball of Z_a, once composed with this isomorphism, give $g_a(z) = \frac{a+z}{1+\bar{a}\ z}$, where a and z are in the open unit ball of $C_0(\Omega)$. For $z = \frac{r}{\|a\|}a$, we have $z \in C_0(\Omega)$ and $\|z\| = r$. Hence

$$g_a(z) = \frac{\left(1 + \frac{r}{\|a\|}\right)\ a}{1 + |a|^2\ \frac{r}{\|a\|}} = \frac{r + \|a\|}{\|a\| + r\ |a|^2}\ a.$$

Now, $\|g_a(z)\| = (r+\|a\|)\ \left\| \frac{a}{\|a\|+r\ |a|^2} \right\| = (r+\|a\|)\ \sup_{\omega\in\Omega} \frac{|a|}{\|a\|+r\ |a|^2}(\omega)$. But since $|a| \leq \|a\| \leq 1$ and $r < 1$, it turns out that $\frac{|a|}{\|a\|+r\ |a|^2}$ is an increasing function of $|a|$, that is $\left\| \frac{a}{\|a\|+r\ |a|^2} \right\| = \frac{1}{1+r\ \|a\|}$. This gives

$$\|g_a(z)\| = \frac{\|a\| + \|z\|}{1 + \|a\| \cdot \|z\|}$$

which is what was required.

4 A result for composition operators

The following result is a very well known version of the Schwarz lemma for Banach spaces (cf. [10]).

Lemma 2 *Let X and Y be Banach spaces and $f : B_X \longrightarrow B_Y$ holomorphic with $f(0) = 0$. Then, for all $x \in B_X$,*

$$\|f(x)\|_Y \leq \|x\|_X.$$

We can now prove a generalization of [8, Theorem 2.3] and [14, Theorem 4.1]. The statement is slightly different from the previous cases but the proof is basically the same, up to technical changes. We include a proof for the sake of completeness.

Theorem 1 *Let X be any Banach space and Z a JB^*-triple. Let v_Z be a norm-radial and non-increasing weight on Z and v_X be a weight on X for which there exists $K > 0$ such that*

if $z \in Z$ and $x \in X$ with $\|z\| \leq \|x\|$, then $v_Z(z) \geq K v_X(x)$.

Then every composition operator $C_\phi : H_{v_Z}(B_Z) \longrightarrow H_{v_X}(B_X)$ is continuous for every holomorphic map $\phi : B_X \to B_Z$ if and only if the function $l(r) := \tilde{v}_Z(z)$ for $\|z\| = 1 - r$, $0 < r < 1$ satisfies $l(s) \leq M l(s/2)$ for s close enough to 0.

Proof First, if $\phi(0) = 0$ then by the general version of the Schwarz Lemma we have $\|\phi(x)\|_Z \leq \|x\|_X$ and C_ϕ is continuous. For each $a \in B_Z$ we have $g_a : B_Z \to B_Z$. If every C_{g_a} is continuous then all C_ϕ are continuous. Indeed, given ϕ, let $a = \phi(0)$ and define $\psi = g_{-a} \circ \phi$. Then $\psi(0) = 0$ and $C_\phi = C_\psi \circ C_{g_a}$ is continuous. Therefore it is enough to prove that $C_{g_a} : H_{v_Z}(B_Z) \to H_{v_Z}(B_Z)$ is continuous for all $a \in B_Z$ if and only if, for all $0 < s < s_0$,

$$l(s) \leq M l(s/2) \tag{2}$$

Assume that all C_{g_a} are continuous. By Proposition 1, for each $a \in B_Z$ we can find $M_a > 0$ such that $\tilde{v}_Z(z) \leq M_a \tilde{v}_Z(g_a(z))$ for all $z \in B_Z$. We also know that $\sup_{\|z\|=r} \|g_a(z)\| = \frac{\|a\|+r}{1+r\|a\|}$. Since v_Z is norm-radial and non-increasing so also is $\tilde{v}_Z$. Hence the previous can be rewritten as

$$l(1-r) \leq M_a l\left(1 - \frac{\|a\| + r}{1 + r\|a\|}\right) = M_a l\left(\frac{(1-r)(1-\|a\|)}{1 + r\|a\|}\right).$$

Now, for $1/2 < r < 1$ we have

$$l\left((1-r)\ \frac{1-\|a\|}{1+\|a\|}\right) \leq l\left(1 - \frac{\|a\|+r}{1+r\|a\|}\right) \leq l\left((1-r)\ \frac{1-\|a\|}{1+\|a\|/2}\right). \quad (3)$$

Let us fix a with $\|a\| = 2/5$ and use the second inequality in (3) to get $l(1-r) \leq M_a l\left(\frac{(1-r)(1-\|a\|)}{1+r\|a\|}\right) \leq M_a l(\frac{1-r}{2})$ for $1/2 < r < 1$. This shows that (2) holds.

Let us assume now that (2) holds. Given any $c > 0$ we can choose $n \in \mathbb{N}$ with $c < 2^n$. If $s < s_0$, then $l(s) \leq K^n\ l(s/c)$. Given any $a \in B_Z$, let us take $c = \frac{1+\|a\|}{1-\|a\|}$ and use the first inequality in (3) to get that there exists $K_a > 0$ such that holds.

$$l(s) \leq K_a l(s/c) \leq K_a l\left(1 - \frac{\|a\| + (1-s)}{1+(1-s)\|a\|}\right)$$

for $s < s_0 \leq 1/2$. Now, for $s_0 \leq t \leq 1$, since l is strictly positive, the mapping $s \rightsquigarrow (l(s))(l(1 - \frac{\|a\|(1-s)}{1+(1-s)\|a\|}))^{-1}$ is well defined and continuous; hence it has a maximum. Thus for any fixed $a \in B_Z$ we can find a constant $M_a > 0$ such that for $0 < r < 1$ and $\|z\| = r$,

$$\tilde{v}_Z(z) \leq M_a\ l\left(1 - \frac{\|a\|+r}{1+r\|a\|}\right) \leq M_a\ \tilde{v}_Z(g_a(z)).$$

Applying Proposition 1, C_{g_a} is continuous.

Several equivalent conditions on a weight v so that l satisfies (2) are given in [11, Lemma 1] for the one-dimensional case. Most of the proofs can be trivially adapted to the infinite dimensional case.

By taking $X = Z$ and $v_X = v_Z$ in Theorem 1 we get

Corollary 1 *Let v be a norm-radial and non-increasing weight on a JB^*-triple Z. Every composition operator C_ϕ on the weighted Banach space $H_v(B_Z)$ is continuous for every self map ϕ on B_Z if and only if the function $l(r) := \tilde{v}(z)$ for $\|z\| = 1-r$, $0 < r < 1$ satisfies $l(s) \leq Ml(s/2)$ for s close enough to 0.*

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