Ecuaciones Algebraicas lineales

- An equation of the form $ax+by+c=0$ or equivalently $ax+by=-c$ is called a linear equation in x and y variables.
- $ax+by+cz=d$ is a linear equation in three variables, x, y, and z.
- Thus, a linear equation in n variables is

  $a_1x_1+a_2x_2+ \dots +a_nx_n = b$
- A solution of such an equation consists of real numbers c_1, c_2, c_3, … , c_n. If you need to work more than one linear equations, a system of linear equations must be solved simultaneously.

In Part Two, we determined the value x that satisfied a single equation, $f(x) = 0$. Now, we deal with the case of determining the values $x_1, x_2, \ldots, x_n$ that simultaneously satisfy a set of equations

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ \vdots \quad & \quad \vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned} \longrightarrow \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Matrices

$$\begin{array}{ccc} & \text{Column 3} & \\ [A] = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{c} \\ \text{Row 2} \\ \\ \\ \end{array} \end{array}$$

a_{ij} = elementos de una matriz

i=número del renglón
j=número de la columna

$$[B] = [b_1 \quad b_2 \quad \cdots \quad b_m]$$

Vector renglón

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

Vector columna

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Matriz cuadrada m=n

Número de ecuaciones

Diagonal principal

Número de incógnitas

Box PT3.1 Special Types of Square Matrices

There are a number of special forms of square matrices that are important and should be noted:

A *symmetric matrix* is one where $a_{ij} = a_{ji}$ for all i's and j's. For example,

$$[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

is a 3 by 3 symmetric matrix.

A *diagonal matrix* is a square matrix where all elements off the main diagonal are equal to zero, as in

$$[A] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{bmatrix}$$

Note that where large blocks of elements are zero, they are left blank.

An *identity matrix* is a diagonal matrix where all elements on the main diagonal are equal to 1, as in

$$[I] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

The symbol $[I]$ is used to denote the identity matrix. The identity matrix has properties similar to unity.

An *upper triangular matrix* is one where all the elements below the main diagonal are zero, as in

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix}$$

A *lower triangular matrix* is one where all elements above the main diagonal are zero, as in

$$[A] = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A *banded matrix* has all elements equal to zero, with the exception of a band centered on the main diagonal:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$$

The above matrix has a bandwidth of 3 and is given a special name—the *tridiagonal matrix*.

Reglas de operaciones con matrices

Addition of two matrices, say, [A] and [B], is accomplished by adding corresponding terms in each matrix. The elements of the resulting matrix [C] are computed,

$$c_{ij} = a_{ij} + b_{ij}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Similarly, the *subtraction* of two matrices, say, [E] minus [F], is obtained by subtracting corresponding terms, as in

$$d_{ij} = e_{ij} - f_{ij}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. It follows directly from the above definitions that addition and subtraction can be performed only between matrices having the same dimensions.

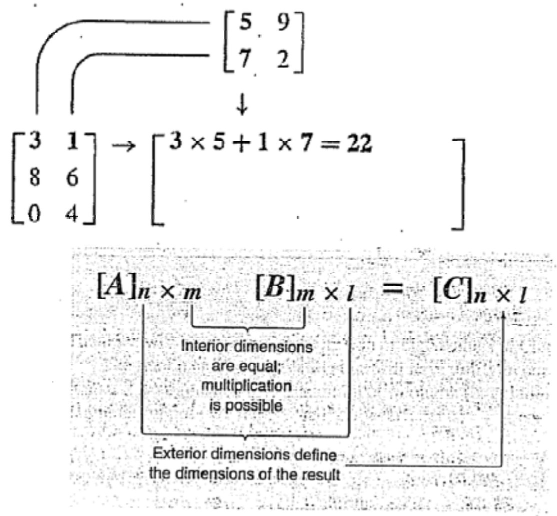
Both addition and subtraction are *commutative:*

$$[A] + [B] = [B] + [A]$$

Addition and subtraction are also *associative*, that is,

$$([A] + [B]) + [C] = [A] + ([B] + [C])$$

The *multiplication* of a matrix $[A]$ by a scalar g is obtained by multiplying every element of $[A]$ by g, as in



$$[D] = g[A] = \begin{bmatrix} ga_{11} & ga_{12} & \cdots & ga_{1m} \\ ga_{21} & ga_{22} & \cdots & ga_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ ga_{n1} & ga_{n2} & \cdots & ga_{nm} \end{bmatrix}$$

The *product* of two matrices is represented as $[C] = [A][B]$, where the elements of $[C]$ are defined as (see Box PT3.2 for a simple way to conceptualize matrix multiplication)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{PT3.2}$$

where n = the column dimension of $[A]$ and the row dimension of $[B]$. That is, the c_{ij} element is obtained by adding the product of individual elements from the ith row of the first matrix, in this case $[A]$, by the jth column of the second matrix $[B]$.

$$[A][B] \neq [B][A]$$

Although multiplication is possible, matrix division is not a defined operation. However, if a matrix $[A]$ is square and nonsingular, there is another matrix $[A]^{-1}$, called the *inverse* of $[A]$, for which

$$[A][A]^{-1} = [A]^{-1}[A] = [I] \tag{PT3.3}$$

Two other matrix manipulations that will have utility in our discussion are the transpose and the trace of a matrix. The transpose of a matrix involves transforming its rows into columns and its columns into rows. For example, for the 4×4 matrix,

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

the transpose, designated $[A]^T$, is defined as

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

In other words, the element a_{ij} of the transpose is equal to the a_{ji} element of the original matrix.

The *transpose* has a variety of functions in matrix algebra. One simple advantage is that it allows a column vector to be written as a row. For example, if

$$\{C\} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}$$

then

$$\{C\}^T = \lfloor c_1 \quad c_2 \quad c_3 \quad c_4 \rfloor$$

The *trace* of a matrix is the sum of the elements on its principal diagonal. It is designated as tr $[A]$ and is computed as

$$\operatorname{tr}[A] = \sum_{i=1}^{n} a_{ii}$$

The trace will be used in our discussion of eigenvalues in Chap. 27.

The final matrix manipulation that will have utility in our discussion is *augmentation.* A matrix is augmented by the addition of a column (or columns) to the original matrix. For example, suppose we have a matrix of coefficients:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We might wish to augment this matrix $[A]$ with an identity matrix (recall Box PT3.1) to yield a 3-by-6-dimensional matrix:

$$[A] = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Such an expression has utility when we must perform a set of identical operations on two matrices. Thus, we can perform the operations on the single augmented matrix rather than on the two individual matrices.

Representación de ecuaciones algebraicas lineales en forma matricial

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$[A]^{-1}[A]\{X\} = [A]^{-1}\{B\}$$

$$\{X\} = [A]^{-1}\{B\}$$

Solving for X

$$
[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
$$

$$[A]\{X\} = \{B\}$$

$$\{X\}^T = \lfloor x_1 \quad x_2 \quad \cdots \quad x_n \rfloor$$

$$\{B\}^T = \lfloor b_1 \quad b_2 \quad \cdots \quad b_n \rfloor$$

Noncomputer Methods for Solving Systems of Equations

- For small number of equations ($n \leq 3$) linear equations can be solved readily by simple techniques such as “method of elimination.”
- Linear algebra provides the tools to solve such systems of linear equations.
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.

Gauss Elimination

Chapter 9

Solving Small Numbers of Equations

- There are many ways to solve a system of linear equations:
  - Graphical method
  - Cramer's rule
  - Method of elimination
  - Computer methods

For $n \leq 3$

Graphical Method

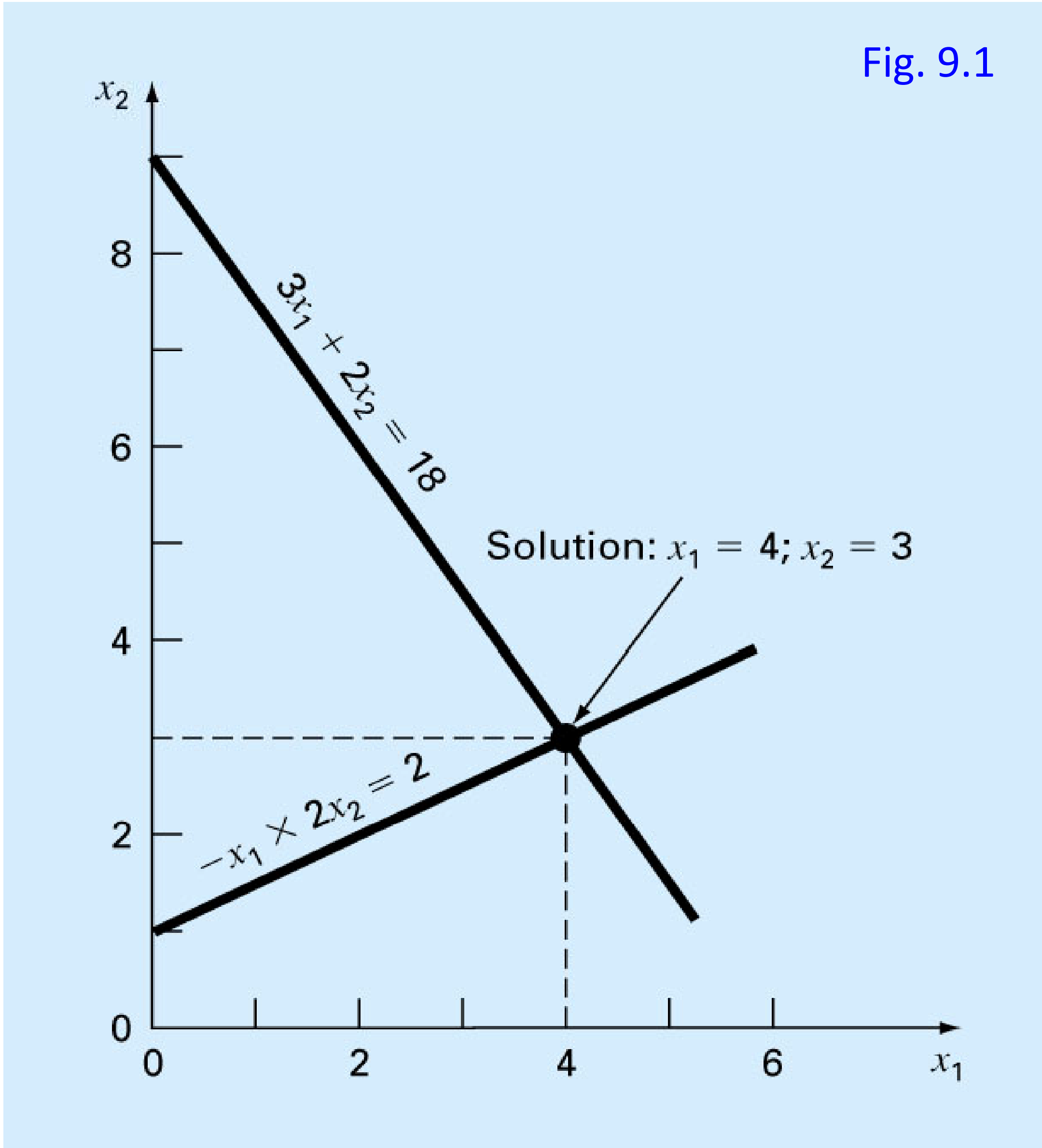
- For two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Solve both equations for $x_{2:}$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$
$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

- Plot x_2 vs. x_1 on rectilinear paper, the intersection of the lines present the solution.



Graphical Method

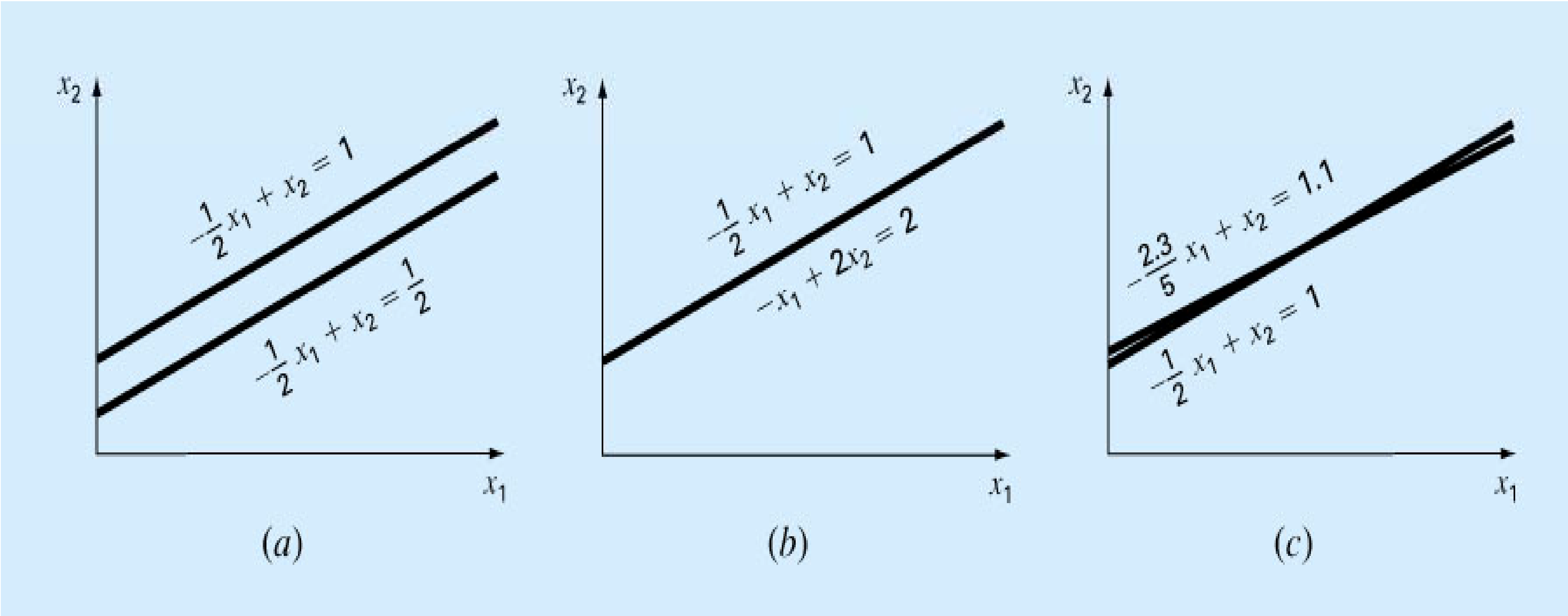
- Or equate and solve for x_1

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

$$\Rightarrow \left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} - \frac{b_2}{a_{22}} = 0$$

$$\Rightarrow x_1 = -\frac{\left(\frac{b_1}{a_{12}} - \frac{b_2}{a_{22}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)} = \frac{\left(\frac{b_2}{a_{22}} - \frac{b_1}{a_{12}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)}$$

Figure 9.2



No solution

Infinite solutions

Ill-conditioned

(Slopes are too close)

Determinants and Cramer's Rule

- Determinant can be illustrated for a set of three equations:

$$Ax = b$$

- Where A is the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Assuming all matrices are square matrices, there is a number associated with each square matrix A called the determinant, D, of A. (D=det (A)). If [A] is order 1, then [A] has one element:

  $A=[a_{11}]$

  $D=a_{11}$

- For a square matrix of order 2, A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

the determinant is $D= a_{11}\, a_{22}-a_{21}\, a_{12}$

- For a square matrix of order 3, the ***minor*** of an element a_{ij} is the determinant of the matrix of order 2 by deleting row *i* and column *j* of A.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{32}\, a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\, a_{33} - a_{31}\, a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- *Cramer's rule* expresses the solution of a systems of linear equations in terms of ratios of determinants of the array of coefficients of the equations. For example, x_1 would be computed as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

Determinants

Problem Statement. Compute values for the determinants of the systems represented in Figs. 9.1 and 9.2.

Solution. For Fig. 9.1:

$$D = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 3(2) - 2(-1) = 8$$

For Fig. 9.2*a*:

$$D = \begin{vmatrix} -1/2 & 1 \\ -1/2 & 1 \end{vmatrix} = \frac{-1}{2}(1) - 1\left(\frac{-1}{2}\right) = 0$$

For Fig. 9.2*b*:

$$D = \begin{vmatrix} -1/2 & 1 \\ -1 & 2 \end{vmatrix} = \frac{-1}{2}(2) - 1(-1) = 0$$

For Fig. 9.2*c*:

$$D = \begin{vmatrix} -1/2 & 1 \\ -2.3/5 & 1 \end{vmatrix} = \frac{-1}{2}(1) - 1\left(\frac{-2.3}{5}\right) = -0{,}04$$

Cramer's Rule

Problem Statement. Use Cramer's rule to solve

$$0.3x_1 + 0.52x_2 + x_3 = -0.01$$
$$0.5x_1 + x_2 + 1.9x_3 = 0.67$$
$$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$$

Solution. The determinant D can be written as [Eq. (9.2

$$D = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

The minors are [Eq. (9.3)]

$$A_1 = \begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} = 1(0.5) - 1.9(0.3) = -0.07$$

$$A_2 = \begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} = 0.5(0.5) - 1.9(0.1) = 0.06$$

$$A_3 = \begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = 0.5(0.3) - 1(0.1) = 0.05$$

These can be used to evaluate the determinant, as in [Eq. (9.4)]

$$D = 0.3(-0.07) - 0.52(0.06) + 1(0.05) = -0.0022$$

Applying Eq. (9.5), the solution is

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

Method of Elimination

- The basic strategy is to successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.
- The elimination of unknowns can be extended to systems with more than two or three equations; however, the method becomes extremely tedious to solve by hand.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = b_1 a_{21}$$
$$a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = b_2 a_{11}$$

$$a_{22}a_{11}x_2 - a_{12}a_{21}x_2 = b_2 a_{11} - b_1 a_{21}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Relación con Cramer

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

Naive Gauss Elimination

- Extension of *method of elimination* to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back substitute.
- As in the case of the solution of two equations, the technique for *n* equations consists of two phases:
  - Forward elimination of unknowns
  - Back substitution

Fig. 9.3

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3 / a''_{33}$$

$$x_2 = (c'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

Generalizando

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots & \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Multiplicando ec 1

Elemento pivote

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \cdots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Restando ec2 de la nueva ec1

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

Reescribiendo ec anterior

$$a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \cdots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad\qquad \vdots & \\
a'_{n2}x_2 + a'_{n3}x_3 + \cdots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

a_{32}'/a_{22}' = nuevo elemento pivote

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad\qquad \vdots & \\
a''_{n3}x_3 + \cdots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad & \quad \vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution. Equation (9.15*d*) can now be solved for x_n:

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \qquad \text{for } i = n-1, n-2, \ldots, 1$$

```
DOFOR k = 1, n − 1
 DOFOR i = k + 1, n
   factor = a_i,k / a_k,k
   DOFOR j = k + 1 to n
     a_i,j = a_i,j − factor · a_k,j
   END DO
   b_i = b_i − factor · b_k
 END DO
END DO
x_n = b_n / a_n,n
DOFOR i = n − 1, 1, −1
  sum = b_i
  DOFOR j = i + 1, n
    sum = sum − a_i,j · x_j
  END DO
  x_i = b_i / a_i,i
END DO
```

Problem Statement. Use Gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \tag{E9.5.1}$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \tag{E9.5.2}$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \tag{E9.5.3}$$

Carry six significant figures during the computation.

Solution. The first part of the procedure is forward elimination. Multiply Eq. (E9.5.1) by $(0.1)/3$ and subtract the result from Eq. (E9.5.2) to give

$$7.00333x_2 - 0.293333x_3 = -19.5617$$

Then multiply Eq. (E9.5.1) by $(0.3)/3$ and subtract it from Eq. (E9.5.3) to eliminate x_1. After these operations, the set of equations is

$$3x_1 \quad - 0.1x_2 \quad - 0.2x_3 = 7.85 \tag{E9.5.4}$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \tag{E9.5.5}$$

$$-0.190000x_2 + 10.0200x_3 = 70.6150 \tag{E9.5.6}$$

To complete the forward elimination, x_2 must be removed from Eq. (E9.5.6). To accomplish this, multiply Eq. (E9.5.5) by $-0.190000/7.00333$ and subtract the result from Eq. (E9.5.6). This eliminates x_2 from the third equation and reduces the system to an upper triangular form, as in

$$3x_1 \quad - 0.1x_2 \quad - 0.2x_3 = 7.85 \tag{E9.5.7}$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \tag{E9.5.8}$$

$$10.0120x_3 = 70.0843 \tag{E9.5.9}$$

We can now solve these equations by back substitution. First, Eq. (E9.5.9) can be solved for

$$x_3 = \frac{70.0843}{10.0120} = 7.0000 \tag{E9.5.10}$$

This result can be back-substituted into Eq. (E9.5.8):

$$7.00333x_2 - 0.293333(7.0000) = -19.5617$$

which can be solved for

$$x_2 = \frac{-19.5617 + 0.293333(7.0000)}{7.00333} = -2.50000 \tag{E9.5.11}$$

Finally, Eqs. (E9.5.10) and (E9.5.11) can be substituted into Eq. (E9.5.4):

$$3x_1 - 0.1(-2.50000) - 0.2(7.0000) = 7.85$$

which can be solved for

$$x_1 = \frac{7.85 + 0.1(-2.50000) + 0.2(7.0000)}{3} = 3.00000$$

The results are identical to the exact solution of $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$. This can be verified by substituting the results into the original equation set

$$3(3) - 0.1(-2.5) - 0.2(7) = 7.85$$
$$0.1(3) + 7(-2.5) - 0.3(7) = -19.3$$
$$0.3(3) - 0.2(-2.5) + 10(7) = 71.4$$

Pitfalls of Elimination Methods

- *Division by zero*. It is possible that during both elimination and back-substitution phases a division by zero can occur.
- *Round-off errors*.
- *Ill-conditioned systems*. Systems where small changes in coefficients result in large changes in the solution. Alternatively, it happens when two or more equations are nearly identical, resulting a wide ranges of answers to approximately satisfy the equations. Since round off errors can induce small changes in the coefficients, these changes can lead to large solution errors.

- *Singular systems*. When two equations are identical, we would loose one degree of freedom and be dealing with the impossible case of n-1 equations for n unknowns. For large sets of equations, it may not be obvious however. The fact that the determinant of a singular system is zero can be used and tested by computer algorithm after the elimination stage. If a zero diagonal element is created, calculation is terminated.

Techniques for Improving Solutions

- Use of more significant figures.
- *Pivoting*. If a pivot element is zero, normalization step leads to division by zero. The same problem may arise, when the pivot element is close to zero. Problem can be avoided:
  - *Partial pivoting*. Switching the rows so that the largest element is the pivot element.
  - *Complete pivoting*. Searching for the largest element in all rows and columns then switching.

Ill-Conditioned Systems

Problem Statement. Solve the following system:

$$x_1 + 2x_2 = 10 \tag{E9.6.1}$$

$$1.1x_1 + 2x_2 = 10.4 \tag{E9.6.2}$$

Then, solve it again, but with the coefficient of x_1 in the second equation modified slightly to 1.05.

Solution.

$$x_1 = \frac{2(10) - 2(10.4)}{1(2) - 2(1.1)} = 4$$

Cramer o sustituciòn

$$x_2 = \frac{1(10.4) - 1.1(10)}{1(2) - 2(1.1)} = 3$$

However, with the slight change of the coefficient a_{21} from 1.1 to 1.05, the result is changed dramatically to

$$x_1 = \frac{2(10) - 2(10.4)}{1(2) - 2(1.05)} = 8$$

$$x_2 = \frac{1(10.4) - 1.1(10)}{1(2) - 2(1.05)} = 1$$

Notice that the primary reason for the discrepancy between the two results is that the denominator represents the difference of two almost-equal numbers. As illustrated previously in Sec. 3.4.2, such differences are highly sensitive to slight variations in the numbers being manipulated.

At this point, you might suggest that substitution of the results into the original equations would alert you to the problem. Unfortunately, for ill-conditioned systems this is often not the case. Substitution of the erroneous values of $x_1 = 8$ and $x_2 = 1$ into Eqs. (E9.6.1) and (E9.6.2) yields

$$8 + 2(1) = 10 = 10$$

$$1.1(8) + 2(1) = 10.8 \cong 10.4$$

Therefore, although $x_1 = 8$ and $x_2 = 1$ is not the true solution to the original problem, the error check is close enough to possibly mislead you into believing that your solutions are adequate.

Effect of Scale on the Determinant

Problem Statement. Evaluate the determinant of the following systems:

(a) From Example 9.1:

$$3x_1 + 2x_2 = 18 \quad \text{(E9.7.1)}$$

$$-x_1 + 2x_2 = 2 \quad \text{(E9.7.2)}$$

(b) From Example 9.6:

$$x_1 + 2x_2 = 10 \quad \text{(E9.7.3)}$$

$$1.1x_1 + 2x_2 = 10.4 \quad \text{(E9.7.4)}$$

(c) Repeat (*b*) but with the equations multiplied by 10.

Solution.

(a) The determinant of Eqs. (E9.7.1) and (E9.7.2), which are well-conditioned, is

$$D = 3(2) - 2(-1) = 8$$

(b) The determinant of Eqs. (E9.7.3) and (E9.7.4), which are ill-conditioned, is

$$D = 1(2) - 2(1.1) = -0.2$$

(c) The results of (*a*) and (*b*) seem to bear out the contention that ill-conditioned systems have near-zero determinants. However, suppose that the ill-conditioned system in (*b*) is multiplied by 10 to give

$$10x_1 + 20x_2 = 100$$

$$11x_1 + 20x_2 = 104$$

The multiplication of an equation by a constant has no effect on its solution. In addition, it is still ill-conditioned. This can be verified by the fact that multiplying by a constant has no effect on the graphical solution. However, the determinant is dramatically affected:

$$D = 10(20) - 20(11) = -20$$

Not only has it been raised two orders of magnitude, but it is now over twice as large as the determinant of the well-conditioned system in (*a*).

Scaling

Problem Statement. Scale the systems of equations in Example 9.7 to a maximum value of 1 and recompute their determinants.

Solution.

(a) For the well-conditioned system, scaling results in

$$\begin{aligned} x_1 + 0.667x_2 &= 6 \\ -0.5x_1 + \quad\quad x_2 &= 1 \end{aligned}$$

for which the determinant is

$$D = 1(1) - 0.667(-0.5) = 1.333$$

(b) For the ill-conditioned system, scaling gives

$$\begin{aligned} 0.5x_1 + x_2 &= 5 \\ 0.55x_1 + x_2 &= 5.2 \end{aligned}$$

for which the determinant is

$$D = 0.5(1) - 1(0.55) = -0.05$$

(c) For the last case, scaling changes the system to the same form as in (*b*) and the determinant is also -0.05. Thus, the scale effect is removed.

Determinant Evaluation Using Gauss Elimination

The method is based on the fact that the determinant of a triangular matrix can be simply computed as the product of its diagonal elements:

$$D = a_{11}a_{22}a_{33} \cdots a_{nn} \tag{B9.1.1}$$

The validity of this formulation can be illustrated for a 3 by 3 system:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}$$

where the determinant can be evaluated as [recall Eq. (9.4)]

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix}$$

or, by evaluating the minors (that is, the 2 by 2 determinants),

$$D = a_{11}a_{22}a_{33} - a_{12}(0) + a_{13}(0) = a_{11}a_{22}a_{33}$$

Recall that the forward-elimination step of Gauss elimination results in an upper triangular system. Because the value of the determinant is not changed by the forward-elimination process, the determinant can be simply evaluated at the end of this step via

$$D = a_{11}a'_{22}a''_{33} \cdots a^{(n-1)}_{nn} \tag{B9.1.2}$$

There is a slight modification to the above approach when the program employs partial pivoting (Sec. 9.4.2). For this case, the determinant changes sign every time a row is pivoted. One way to represent this is to modify Eq. (B9.1.2):

$$D = a_{11}a'_{22}a''_{33} \cdots a_{nn}^{(n-1)}(-1)^p \tag{B9.1.3}$$

where p represents the number of times that rows are pivoted.

Partial Pivoting

Problem Statement. Use Gauss elimination to solve

$$0.0003x_1 + 3.0000x_2 = 2.0001$$
$$1.0000x_1 + 1.0000x_2 = 1.0000$$

Note that in this form the first pivot element, $a_{11} = 0.0003$, is very close to zero. Then repeat the computation, but partial pivot by reversing the order of the equations. The exact solution is $x_1 = 1/3$ and $x_2 = 2/3$.

Solution. Multiplying the first equation by $1/(0.0003)$ yields

$$x_1 + 10{,}000x_2 = 6667$$

which can be used to eliminate x_1 from the second equation:

$$-9999x_2 = -6666$$

which can be solved for

$$x_2 = \frac{2}{3}$$

This result can be substituted back into the first equation to evaluate x_1:

$$x_1 = \frac{\boxed{2.0001 - 3(2/3)}}{0.0003}$$

Casi cero !!!

Depende del numero de cifras significativas

Note how the solution for x_1 is highly dependent on the number of significant figures. This is because in Eq. (E9.9.1), we are subtracting two almost-equal numbers. On the other hand, if the equations are solved in reverse order, the row with the larger pivot element is normalized. The equations are

$$1.0000x_1 + 1.0000x_2 = 1.0000$$
$$0.0003x_1 + 3.0000x_2 = 2.0001$$

Elimination and substitution yield $x_2 = 2/3$. For different numbers of significant figures, x_1 can be computed from the first equation, as in

$$x_1 = \frac{1 - (2/3)}{1} \tag{E9.9.2}$$

```
p = k
big = |a_k,k|
DOFOR ii = k+1, n
  dummy = |a_ii,k|
  IF (dummy > big)
    big = dummy
    p = ii
  END IF
END DO
IF (p ≠ k)
  DOFOR jj = k, n
    dummy = a_p,jj
    a_p,jj = a_k,jj
    a_k,jj = dummy
  END DO
  dummy = b_p
  b_p = b_k
  b_k = dummy
END IF
```

FIGURE 9.5
Pseudocode to implement partial pivoting.

SCALING

Effect of Scaling on Pivoting and Round-Off

Problem Statement.

(a) Solve the following set of equations using Gauss elimination and a pivoting strategy:

$$2x_1 + 100{,}000x_2 = 100{,}000$$
$$x_1 + \qquad x_2 = 2$$

(b) Repeat the solution after scaling the equations so that the maximum coefficient in each row is 1.

(c) Finally, use the scaled coefficients to determine whether pivoting is necessary. However, actually solve the equations with the original coefficient values. For all cases, retain only three significant figures. Note that the correct answers are $x_1 = 1.00002$ and $x_2 = 0.99998$ or, for three significant figures, $x_1 = x_2 = 1.00$.

Solution.

(a) Without scaling, forward elimination is applied to give

$$2x_1 + 100{,}000x_2 = 100{,}000$$
$$-50{,}000x_2 = -50{,}000$$

which can be solved by back substitution for

$$x_2 = 1.00$$
$$x_1 = 0.00$$

Although x_2 is correct, x_1 is 100 percent in error because of round-off.

(b) Scaling transforms the original equations to

$$0.00002x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Therefore, the rows should be pivoted to put the greatest value on the diagonal.

$$x_1 + x_2 = 2$$
$$0.00002x_1 + x_2 = 1$$

Forward elimination yields

$$x_1 + x_2 = 2$$
$$x_2 = 1.00$$

which can be solved for

$$x_1 = x_2 = 1$$

Thus, scaling leads to the correct answer.

(c) The scaled coefficients indicate that pivoting is necessary. We therefore pivot but retain the original coefficients to give

$$x_1 + x_2 = 2$$
$$2x_1 + 100{,}000x_2 = 100{,}000$$

Forward elimination yields

$$x_1 + x_2 = 2$$
$$100{,}000x_2 = 100{,}000$$

which can be solved for the correct answer: $x_1 = x_2 = 1$. Thus, scaling was useful in determining whether pivoting was necessary, but the equations themselves did not require scaling to arrive at a correct result.

```
SUB Gauss (a, b, n, x, tol, er)
  DIMENSION s(n)
  er = 0
  DOFOR i = 1, n
    s_i = ABS(a_i,1)
    DOFOR j = 2, n
      IF ABS(a_i,j)>s_i THEN s_i = ABS(a_i,j)
    END DO
  END DO
  CALL Eliminate(a, s, n, b, tol, er)
  IF er ≠ −1 THEN
     CALL Substitute(a, n, b, x)
  END IF
END Gauss
```

```
SUB Eliminate (a, s, n, b, tol, er)
  DOFOR k = 1, n − 1
    CALL Pivot (a, b, s, n, k)
    IF ABS (a_k,k/s_k) < tol THEN
      er = −1
      EXIT DO
    END IF
    DOFOR i = k + 1, n
      factor = a_i,k/a_k,k
      DOFOR j = k + 1, n
        a_i,j = a_i,j − factor*a_k,j
      END DO
      b_i = b_i − factor * b_k
    END DO
  END DO
  IF ABS(a_k,k/s_k) < tol THEN er = −1
END Eliminate
```

```
SUB Pivot (a, b, s, n, k)
  p = k
  big = ABS(a_k,k/s_k)
  DOFOR ii = k + 1, n
    dummy = ABS(a_ii,k/s_ii)
    IF dummy > big THEN
      big = dummy
      p = ii
    END IF
  END DO
  IF p ≠ k THEN
    DOFOR jj = k, n
      dummy = a_p,jj
      a_p,jj = a_k,jj
      a_k,jj = dummy
    END DO
    dummy = b_p
    b_p = b_k
    b_k = dummy
    dummy = s_p
    s_p = s_k
    s_k = dummy
  END IF
END pivot
```

```
SUB Substitute (a, n, b, x)
  x_n = b_n/a_n,n
  DOFOR i = n − 1, 1, −1
    sum = 0
    DOFOR j = i + 1, n
      sum = sum + a_i,j * x_j
    END DO
    x_i = (b_i − sum) / a_i,i
  END DO
END Substitute
```

Gauss-Jordan

- It is a variation of Gauss elimination. The major differences are:
  - When an unknown is eliminated, it is eliminated from all other equations rather than just the subsequent ones.
  - All rows are normalized by dividing them by their pivot elements.
  - Elimination step results in an identity matrix.
  - Consequently, it is not necessary to employ back substitution to obtain solution.

Gauss-Jordan Method

Problem Statement. Use the Gauss-Jordan technique to solve the same system as in Example 9.5:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution. First, express the coefficients and the right-hand side as an augmented matrix:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Then normalize the first row by dividing it by the pivot element, 3, to yield

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

The x_1 term can be eliminated from the second row by subtracting 0.1 times the first row from the second row. Similarly, subtracting 0.3 times the first row from the third row will eliminate the x_1 term from the third row:

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.293333 & -19.5617 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Next, normalize the second row by dividing it by 7.00333:

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Reduction of the x_2 terms from the first and third equations gives

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{bmatrix}$$

The third row is then normalized by dividing it by 10.0120:

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

Finally, the x_3 terms can be reduced from the first and the second equations to give

$$\begin{bmatrix} 1 & 0 & 0 & 3.0000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

Thus, as depicted in Fig. 9.8, the coefficient matrix has been transformed to the identity matrix, and the solution is obtained in the right-hand-side vector. Notice that no back substitution was required to obtain the solution.

PROBLEMS

9.1

(a) Write the following set of equations in matrix form:

$$50 = 5x_3 + 2x_2$$
$$10 - x_1 = x_3$$
$$3x_2 + 8x_1 = 20$$

(b) Write the transpose of the matrix of coefficients.

9.2 A number of matrices are defined as

$$[A] = \begin{bmatrix} 4 & 7 \\ 1 & 2 \\ 5 & 6 \end{bmatrix} \qquad [B] = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\{C\} = \begin{Bmatrix} 3 \\ 6 \\ 1 \end{Bmatrix} \qquad [D] = \begin{bmatrix} 9 & 4 & 3 & -6 \\ 2 & -1 & 7 & 5 \end{bmatrix}$$

$$[E] = \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}$$

$$[F] = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix} \qquad \lfloor G \rfloor = \lfloor 7 \quad 6 \quad 4 \rfloor$$

Answer the following questions regarding these matrices:

(a) What are the dimensions of the matrices?

(b) Identify the square, column, and row matrices.

(c) What are the values of the elements: a_{12}, b_{23}, d_{32}, e_{22}, f_{12}, g_{12}?

(d) Perform the following operations:

(1) $[E] + [B]$	(3) $[B] - [E]$
(2) $[A] + [F]$	(4) $7 \times [B]$
(5) $[E] \times [B]$	(9) $[A] \times \{C\}$
(6) $\{C\}^T$	(10) $[I] \times [B]$
(7) $[B] \times [A]$	(11) $[E]^T[E]$
(8) $[D]^T$	(12) $\{C\}^T\{C\}$

9.3 Three matrices are defined as

$$[A] = \begin{bmatrix} 1 & 6 \\ 3 & 10 \\ 7 & 4 \end{bmatrix} \qquad [B] = \begin{bmatrix} 1 & 3 \\ 0.5 & 2 \end{bmatrix} \qquad [C] = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

(a) Perform all possible multiplications that can be computed between pairs of these matrices.

(b) Use the method in Box PT3.2 to justify why the remaining pairs cannot be multiplied.

(c) Use the results of (a) to illustrate why the order of multiplication is important.

9.4 Use the graphical method to solve

$$4x_1 - 8x_2 = -24$$
$$x_1 + 6x_2 = 34$$

9.8 Given the equations

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

(a) Solve by naive Gauss elimination. Show all steps of the computation.

(b) Substitute your results into the original equations to check your answers.

9.9 Use Gauss elimination to solve:

$$8x_1 + 2x_2 - 2x_3 = -2$$
$$10x_1 + 2x_2 + 4x_3 = 4$$
$$12x_1 + 2x_2 + 2x_3 = 6$$

Employ partial pivoting and check your answers by substituting them into the original equations.

9.10 Given the system of equations

$$-3x_2 + 7x_3 = 2$$

Check your results by substituting them back into the equations.

9.5 Given the system of equations

$$-1.1x_1 + 10x_2 = 120$$

$$-2x_1 + 17.4x_2 = 174$$

(a) Solve graphically and check your results by substituting them back into the equations.
(b) On the basis of the graphical solution, what do you expect regarding the condition of the system?
(c) Compute the determinant.
(d) Solve by the elimination of unknowns.

9.6 For the set of equations

$$2x_2 + 5x_3 = 9$$

$$2x_1 + x_2 + x_3 = 9$$

$$3x_1 + x_2 = 10$$

(a) Compute the determinant.
(b) Use Cramer's rule to solve for the x's.
(c) Substitute your results back into the original equation to check your results.

9.7 Given the equations

$$0.5x_1 - x_2 = -9.5$$

$$1.02x_1 - 2x_2 = -18.8$$

(a) Solve graphically.
(b) Compute the determinant.
(c) On the basis of (a) and (b), what would you expect regarding the system's condition?
(d) Solve by the elimination of unknowns.
(e) Solve again, but with a_{11} modified slightly to 0.52. Interpret your results.

$$x_1 + 2x_2 - x_3 = 3$$

$$5x_1 - 2x_2 = 2$$

(a) Compute the determinant.
(b) Use Cramer's rule to solve for the x's.
(c) Use Gauss elimination with partial pivoting to solve for the x's.
(d) Substitute your results back into the original equations to check your solution.

9.11 Given the equations

$$2x_1 - 6x_2 - x_3 = -38$$

$$-3x_1 - x_2 + 7x_3 = -34$$

$$-8x_1 + x_2 - 2x_3 = -20$$

(a) Solve by Gauss elimination with partial pivoting. Show all steps of the computation.
(b) Substitute your results into the original equations to check your answers.

9.12 Use Gauss-Jordan elimination to solve:

$$2x_1 + x_2 - x_3 = 1$$

$$5x_1 + 2x_2 + 2x_3 = -4$$

$$3x_1 + x_2 + x_3 = 5$$

Do not employ pivoting. Check your answers by substituting them into the original equations.

9.13 Solve:

$$x_1 + x_2 - x_3 = -3$$

$$6x_1 + 2x_2 + 2x_3 = 2$$

$$-3x_1 + 4x_2 + x_3 = 1$$

Descomposición LU e inversión de Matrices

$$[L]\{D\} = \{B\}$$

$$[L]\{[U]\{X\} - \{D\}\} = [A]\{X\} - \{B\}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad [L][U] = [A]$$

$[A]\{X\}=\{B\}$

$[A]\{X\}-\{B\}=0$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$[U]\{X\}-\{D\}=0$

Gauss Elimination

1. *LU decomposition step.* $[A]$ is factored or "decomposed" into lower $[L]$ and upper $[U]$ triangular matrices.
2. *Substitution step.* $[L]$ and $[U]$ are used to determine a solution $\{X\}$ for a right-hand side $\{B\}$. This step itself consists of two steps. First, Eq. (10.8) is used to generate an intermediate vector $\{D\}$ by forward substitution. Then, the result is substituted into Eq. (10.4), which can be solved by back substitution for $\{X\}$.

De la eliminación hacia delante de Gauss tenemos :

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

The first step in Gauss elimination is to multiply row 1 by the factor [recall Eq. (9.13)]

$$f_{21} = \frac{a_{21}}{a_{11}}$$

and subtract the result from the second row to eliminate a_{21}. Similarly, row 1 is multiplied by

$$f_{31} = \frac{a_{31}}{a_{11}}$$

and the result subtracted from the third row to eliminate a_{31}. The final step is to multiply the modified second row by

$$f_{32} = \frac{a'_{32}}{a'_{22}}$$

and subtract the result from the third row to eliminate a'_{32}.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ f_{21} & a'_{22} & a'_{23} \\ f_{31} & f_{32} & a''_{33} \end{bmatrix}$$

Finalmente

$$[A] \to [L][U]$$

where

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

and

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

LU Decomposition with Gauss Elimination

Problem Statement. Derive an *LU* decomposition based on the Gauss elimination performed in Example 9.5.

Solution. In Example 9.5, we solved the matrix

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

After forward elimination, the following upper triangular matrix was obtained:

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

The factors employed to obtain the upper triangular matrix can be assembled into a lower triangular matrix. The elements a_{21} and a_{31} were eliminated by using the factors

$$f_{21} = \frac{0.1}{3} = 0.03333333 \qquad f_{31} = \frac{0.3}{3} = 0.1000000$$

and the element a'_{32} was eliminated by using the factor

$$f_{32} = \frac{-0.19}{7.00333} \; \frac{a'_{32}}{a'_{22}} = -0.0271300$$

Thus, the lower triangular matrix is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix}$$

Consequently, the *LU* decomposition is

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

This result can be verified by performing the multiplication of $[L][U]$ to give

$$[L][U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.0999999 & 7 & -0.3 \\ 0.3 & -0.2 & 9.99996 \end{bmatrix}$$

where the minor discrepancies are due to round-off.

```
SUB Decompose (a, n)
  DOFOR k = 1, n − 1
    DOFOR i = k + 1, n
      factor = a_i,k/a_k,k
      a_i,k = factor
      DOFOR j = k + 1, n
        a_i,j = a_i,j − factor * a_k,j
      END DO
    END DO
  END DO
END Decompose
```

Encontrando ‘d’ aplicando la eliminación hacia adelante pero solo sobre el vector ‘B’

$$d_i = b_i - \sum_{j=1}^{i-1} a_{ij} d_j \qquad \text{for } i = 2, 3, \ldots, n$$

Encontrando ‘X’ aplicando la sustitución hacia atrás

$$x_n = d_n / a_{nn}$$

$$x_i = \frac{d_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \qquad \text{for } i = n-1, n-2, \ldots, 1$$

The Substitution Steps

Problem Statement. Complete the problem initiated in Example by generating the final solution with forward and back substitution.

Solution. As stated above, the intent of forward substitution is to impose the elimination manipulations, that we had formerly applied to $[A]$, on the right-hand-side vector $\{B\}$. Recall that the system being solved in Example 9.5 was

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

A X = B

and that the forward-elimination phase of conventional Gauss elimination resulted in

U X D

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix} \tag{E10.2.1}$$

The forward-substitution phase is implemented by applying Eq. (10.7) to our problem,

L D B

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

or multiplying out the left-hand side,

$$\begin{aligned} d_1 \qquad\qquad\qquad &= 7.85 \\ 0.0333333 d_1 + \qquad d_2 \qquad &= -19.3 \\ 0.1 d_1 - 0.02713 d_2 + d_3 &= 71.4 \end{aligned}$$

$d_1 = 7.85$
$d_2 = 19.5719$
$d_3 = 70.0843$

We can solve the first equation for d_1,

$$d_1 = 7.85$$

which can be substituted into the second equation to solve for

$$d_2 = -19.3 - 0.0333333(7.85) = -19.5617$$

Both d_1 and d_2 can be substituted into the third equation to give

$$d_3 = 71.4 - 0.1(7.85) + 0.02713(-19.5617) = 70.0843$$

Thus,

$$\{D\} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

which is identical to the right-hand side of Eq. (E10.2.1).

This result can then be substituted into Eq. (10.4), $[U]\{X\} = \{D\}$, to give

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

which can be solved by back substitution (see Example 9.5 for details) for the final solution,

$$\{X\} = \begin{Bmatrix} 3 \\ -2.5 \\ 7.00003 \end{Bmatrix}$$

The following is pseudocode for a subroutine to implement both substitution phases:

```
SUB Substitute (a, n, b, x)
  'forward substitution
  DOFOR i = 2, n
    sum = b_i
    DOFOR j = 1, i - 1
      sum = sum - a_i,j * b_j
    END DO
    b_i = sum
  END DO
  'back substitution
  x_n = b_n / a_n,n
  DOFOR i = n - 1, 1, -1
    sum = 0
    DOFOR j = i + 1, n
      sum = sum + a_i,j * x_j
    END DO
    x_i = (b_i - sum)/a_i,i
  END DO
END Substitute
```

```
SUB Ludecomp (a, b, n, tol, x, er)
  DIM on, sn
  er = 0
  CALL Decompose(a, n, tol, o, s, er)
  IF er <> -1 THEN
    CALL Substitute(a, o, n, b, x)
  END IF
END Ludecomp

SUB Decompose (a, n, tol, o, s, er)
  DOFOR i = 1, n
    oi = i
    si = ABS(ai,1)
    DOFOR j = 2, n
      IF ABS(ai,j)>si THEN si = ABS(ai,j)
    END DO
  END DO
  DOFOR k = 1, n - 1
    CALL Pivot(a, o, s, n, k)
    IF ABS(ao(k),k / so(k)) < tol THEN
      er = -1
      PRINT ao(k),k / so(k)
      EXIT DO
    END IF
    DOFOR i = k + 1, n
      factor = ao(i),k / ao(k),k
      ao(i),k = factor
      DOFOR j = k + 1, n
        ao(i),j = ao(i),j - factor * ao(k),j
      END DO
    END DO
  END DO
  IF ABS(ao(k),k / so(k)) < tol THEN
    er = -1
    PRINT ao(k),k / so(k)
  END IF
END Decompose

SUB Pivot (a, o, s, n, k)
  p = k
  big = ABS(ao(k),k / so(k))
  DOFOR ii = k + 1, n
    dummy = ABS(ao(ii),k / so(ii))
    IF dummy > big THEN
      big = dummy
      p = ii
    END IF
  END DO
  dummy = op
  op = ok
  ok = dummy
END Pivot

SUB Substitute (a, o, n, b, x)
  DOFOR i = 2, n
    sum = bo(i)
    DOFOR j = 1, i - 1
      sum = sum - ao(i),j * bo(j)
    END DO
    bo(i) = sum
  END DO
  xn = bo(n) / ao(n),n
  DOFOR i = n - 1, 1, -1
    sum = 0
    DOFOR j = i + 1, n
      sum = sum + ao(i),j * xj
    END DO
    xi = (bo(i) - sum) / ao(i),i
  END DO
END Substitute
```

FIGURE 10.2
Pseudocode for an *LU* decomposition algorithm.

Matriz Inversa

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

The inverse can be computed in a column-by-column fashion by generating solutions with unit vectors as the right-hand-side constants. For example, if the right-hand-side constant has a 1 in the first position and zeros elsewhere,

$$\{b\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

the resulting solution will be the *first* column of the matrix inverse. Similarly, if a unit vector with a 1 at the second row is used

$$\{b\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

the result will be the *second* column of the matrix inverse.

The best way to implement such a calculation is with the *LU* decomposition algorithm described at the beginning of this chapter. Recall that one of the great strengths of *LU* decomposition is that it provides a very efficient means to evaluate multiple right-hand-side vectors. Thus, it is ideal for evaluating the *multiple* unit vectors needed to compute the inverse.

Matrix Inversion

~~Problem Statement~~. Employ *LU* decomposition to determine the matrix inverse for the system from Example 10.2.

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

Recall that the decomposition resulted in the following lower and upper triangular matrices:

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix}$$

Solution. The first column of the matrix inverse can be determined by performing the forward-substitution solution procedure with a unit vector (with 1 in the first row) as the right-hand-side vector. Thus, Eq. (10.8), the lower-triangular system, can be set up as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

and solved with forward substitution for $\{D\}^T = \lfloor 1 \quad -0.03333 \quad -0.1009 \rfloor$. This vector can then be used as the right-hand side of Eq. (10.3),

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.03333 \\ -0.1009 \end{Bmatrix}$$

which can be solved by back substitution for $\{X\}^T = \lfloor 0.33249 \quad -0.00518 \quad -0.01008 \rfloor$, which is the first column of the matrix,

$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0 & 0 \\ -0.00518 & 0 & 0 \\ -0.01008 & 0 & 0 \end{bmatrix}$$

To determine the second column, Eq. (10.8) is formulated as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

This can be solved for $\{D\}$, and the results are used with Eq. (10.3) to determine $\{X\}^T = \lfloor 0.004944 \quad 0.142903 \quad 0.00271 \rfloor$, which is the second column of the matrix,

$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0.004944 & 0 \\ -0.00518 & 0.142903 & 0 \\ -0.01008 & 0.00271 & 0 \end{bmatrix}$$

Finally, the forward- and back-substitution procedures can be implemented with $\{B\}^T = \lfloor 0 \quad 0 \quad 1 \rfloor$ to solve for $\{X\}^T = \lfloor 0.006798 \quad 0.004183 \quad 0.09988 \rfloor$, which is the final column of the matrix,

$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0.004944 & 0.006798 \\ -0.00518 & 0.142903 & 0.004183 \\ -0.01008 & 0.00271 & 0.09988 \end{bmatrix}$$

The validity of this result can be checked by verifying that $[A][A]^{-1} = [I]$.

Homework

10.2 (a) Use naive Gauss elimination to decompose the following system according to the description in Sec. 10.2.

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

Then, multiply the resulting [L] and [U] matrices to determine that [A] is produced. (**b**) Use LU decomposition to solve the system. Show all the steps in the computation. (**c**) Also solve the system for an alternative right-hand-side vector: $\{B\}^T = \lfloor 12 \;\; 18 \;\; -6 \rfloor$.

10.3

(a) Solve the following system of equations by LU decomposition without pivoting

$$8x_1 + 4x_2 - x_3 = 11$$
$$-2x_1 + 5x_2 + x_3 = 4$$
$$2x_1 - x_2 + 6x_3 = 7$$

(b) Determine the matrix inverse. Check your results by verifying that $[A][A]^{-1} = [I]$.

10.4 Solve the following system of equations using LU decomposition with partial pivoting:

$$2x_1 - 6x_2 - x_3 = -38$$
$$-3x_1 - x_2 + 7x_3 = -34$$
$$-8x_1 + x_2 - 2x_3 = -20$$