Mathematical Induction

José Espinosa
http://www.geocities.com/jespinos57/induction.htm

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1 Mathematical Induction Problems.

1. Let :
$$F(n)=\sum_{k=1}^{p-1}k^{n(p-1)+1}-\frac{n(n-1)}{2}\sum_{k=1}^{p-1}(k^{2p-1}-3k^2)-\frac{p(p-1)(n(p-1)+1))}{2}$$
Prove by induction that $F(n)$ is divisible by p^3, for all integers $n \geq 0$ where p is a prime number > 2.

2. Use mathematical induction to prove the following:
$1+2^{2n}+3^{2n}+2((-1)^{FIB(n)}+1)$ Is divisible by 7 for all integers $n>0$.
$FIB(n+2)=FIB(n+1)+FIB(n)$;$FIB(1)=1$,$FIB(2)=1$Fibonacci sequence.

3. Prove the following in more ways than one :
$2(2^{2n}+5^{2n}+6^{2n})+3(-1)^{n+1}((-1)^{FIB(n)}+1)$ Is divisible by 13 for all integers $n>0$.
$FIB(n+2)=FIB(n+1)+FIB(n)$; $FIB(1)=1$,$FIB(2)=1$ Fibonacci sequence.

4. We guess that:$\sum_{i=1}^{m}a_i$ and that;$\sum_{i=1}^{m}{a_i}^{kbc}$ are divisible by $(bc)^2$ for all odd numbers k.(b and c are both odd primes , $b<c$ and $(c-1)$ is not divisible by b , the a_i are relatively prime with respect to b and c).
Let :
$$F(n)=\sum_{i=1}^{m}a_i^{1+(b-1)(c-1)n}$$
Prove by induction that :
$F(n)$ is divisible by $(bc)^2$,for all integers $n \geq 0$.

5. Let a,b,c be three positive integers where $c=a+b$. Let p be an odd factor of $a^2+b^2+c^2$ (p is not divisible by 3 for part 3 and part 4). Prove by induction that for all integers $n>0$:
   (a) $(a^{6n-4}+b^{6n-4}+c^{6n-4})$ is divisible by p.
   (b) $(a^{6n-2}+b^{6n-2}+c^{6n-2})$ is divisible by p^2.
   (c) $(a^{2^n}+b^{2^n}+c^{2^n})$ is divisible by p.
   (d) $(a^{4^n}+b^{4^n}+c^{4^n})$ is divisible by p^2.

6. Prove that for all integers $n>1$:
   (a) $\sum_{i=1}^{n}F(i)^2=F(n)F(n+1)-5$
   (b) $F(n)^2+F(n+1)^2=F(2n+4)-F(2n-3)$

$F(1) = 1,\ F(2) = 6\ F(n) = F(n-1) + F(n-2)$

7. Let $(2p+1)$ be a prime number where p is an odd number > 1. Prove by math induction that for all integers $n > 0$:

$$\sum_{k=1}^{p} k^{2^n}$$

Is divisible by $(2p+1)$.Prove it in more ways than one.

8. Let $(4p+1)$ be a prime number where p is an odd number > 1. Prove by induction that for all integers $n > 0$:

$$\sum_{k=1}^{p} a_k^{2^n}$$

Is divisible by $(4p+1)$. The a_k are all different, belong to the set of the first $2p$ positive integers and they have the property: $a_k^{2p} - 1$ is divisible by $(4p+1)$. The other members of the set have the property: $b_k^{2p} + 1$ is divisible by $(4p+1)$.

9. Prove that for all integers $n \geq 1$:
$2^{2n-1} + 4^{2n-1} + 9^{2n-1}$ Is not a perfect square.

10. Prove that for every positive integer n: $8^{2^n} - 5^{2^n}$ Is not a perfect square.Prove it in two forms.

11. Let $F(n) = 13^{6n+1} + 30^{6n+1} + 100^{6n+1} + 200^{6n+1}$ y let:

$$G(n) = 2F(n) + 2n(n-2)F(1) - n(n-1)F(2)$$

Prove by induction that for all integers $n \geq 0$: $G(n)$ is divisible by 7^3.

12. Let $f(a)$ be a function from positive integers to positive integers. If $(f(a+b) - kf(a))$ is divisible by p for all positive integers a ,then prove that there exists b_0 such that :
$(f(a+b_0b) - f(a))$ is divisible by p.

13. Prove the following in more ways than one :

$$1 + 2^{4n+2} + 3^{4n+2} + 4^{4n+2} + 5^{4n+2} + 6^{4n+2}$$

Is divisible by 13 for all integers $n \geq 0$.

14. Prove by induction that for all integers $n \geq 0$: $(2(3^{4n+3} + 4^{4n+3}) - 25n^2 + 65n + 68)$ Is divisible by 125.

15. Prove by induction that for each positive integer n: $2^{2^n} + 3^{2^n} + 5^{2^n}$ Is divisible by 19.

16. Let $f(n) = (a-1)f(n-1) + af(n-2)$ and let: $g(n) = f(n+2) + af(n+1) + (a-1)f(n)$. Prove by induction that for all integers $n > 0$:

$$g(n) = (f(1) + f(2))(2a-1)a(n-1)$$

17. Let $f(n) = 3(f(n-1) + f(n-2)) + 1, f(1) = f(2) = 1$
Prove by induction that for all integers $n > 0$:

$$(f(3n) + f(3n+1))$$

Is divisible by 32.

18. Let p be a prime number greater than 5. Let $F(n) = 2^{1+(p-1)n} - 3^{1+(p-1)n} - 5^{1+(p-1)n} + 6^{1+(p-1)n}$ y let: $G(n) = 100F(n) - nF(100)$ Prove by induction that for all integers $n \geq 0 : G(n)$ is divisible by p^2.

19. Let p be a positive integer .Let $F(n)$ be a function from integers to integers. If $F(n)$ satisfies the following :
$$(F(n+3) - 3F(n+2) + 3F(n+1) - F(n)) \equiv 0 \pmod{p^3}$$
Then for all integers $n \geq 0$:
$$F(n) \equiv (\frac{(n-1)(n-2)}{2})F(0) - n(n-2)F(1) + (\frac{n(n-1)}{2})F(2)) \pmod{p^3}$$

20. Let $a(n) = a(n-1) + 2a(n-2) + 1$, $a(1) = a(2) = 1$ Prove by induction that for all integers $n > 0$:
$$a(n) = 2^{n-1} - \frac{((-1)^n + 1)}{2}$$

21. Consider the first n^2 Fibonacci Numbers arranged in an anti clockwise spiral as it is shown next for $n = 3$ and $n = 4$.

5	3	2
8	1	1
13	21	34

987	610	377	233
5	3	2	144
8	1	1	89
13	21	34	55

Notice that for $n = 3(21+1) = 2(8+3)$ and for $n = 4(610+5) = 5(89+34)$.Guess and prove this result for all integers $n > 2$ (not necessarily by induction).

22. What would happen if in problem 21 we change the Fibonacci Numbers by the Lucas Numbers, by the even Fibonacci Numbers,etc.?

23. Let p be a prime number greater that 3 such that divides $a^2 + ab + b^2$ (a relatively prime to b). Show in more ways than one that for all integers $n \geq 0$:
$$a^{4+(p-1)n} + b^{4+(p-1)n} + (a+b)^{4+(p-1)n}$$
Is divisible by p^2.

24. Let $(6p+5)$ be a prime number where p is an integer non negative. Prove by math induction that for all integers $n \geq 0$:
$$\sum_{k=1}^{3p+2} k^{2(3^n)}$$
Is divisible by $(6p+5)$.

25. Let $F(n)$ be the n^{th} Fibonacci Number.Prove in several ways that:
$$F(n)^2 + F(n+1)^2 + F(n+2)^2 + F(n+3)^2 = 3F(2n+3)$$

26. Let $F(n)$ be the n^{th} Fibonacci Number.Prove that for every integer non-negative n:
$$F(5n+3) + F(5n+4)^2$$
Is divisible by 11.

27. Let k be a fixed positive integer and let p be an odd prime number.Let $F(n)$ be a function from integers to integers which satisfies the following congruence:

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}F(n+i)\equiv 0 \pmod{p^k}$$

If $F(a_0)$,$F(a_1)$,...,$F(a_{k-1})$ are divisible by p^k where (a_i-a_j) is not divisible by p for $i\neq j$, then prove that for all integers $n\geq 0$: $F(n)$ is divisible by p^k.

28. Let $F(n)$ be the n^{th} Fibonacci Number .
Let $G_n(a) = 89a^n - F(n)a^{11} - F(n-11)$. Prove that for every integer non-negative n: $G_n(a)$ is divisible by the polynomial a^2-a-1.

29. Let $4k+1$ be a prime number.Prove that for all integers non-negative n:

$$\sum_{i=1}^{2k} i^{4n+2}$$

Is divisible by $4k+1$.

30. Let:

$$F(n)=\sum_{k=1}^{p-1}k^{n(p-1)+1}-\frac{p(p-1)(n(p-1)+1))}{2}$$

And let $G(n) = 500500F(n) - \frac{n(n-1)}{2}F(1001)$. Prove by induction that $G(n)$ is divisible by p^3, for all integers $n\geq 0$ where p is a prime number > 13.

2 Hints and Solutions.

Hint Problem 1 : Use the induction form that is indicated next:

1. The property is true for $n=0$, $n=1$ and $n=2$.
2. If the property is true for n, $(n+1)$and $(n+2)$,this implies that the property is true for $(n+3)$.

Demonstrate the following relation:

$$(F(n+3)-3F(n+2)+3F(n+1)-F(n))\equiv 0 \pmod{p^3}$$

(Use the Fermat's little theorem and the following result: If $g(n)=an^2+bn+c$, is simple to verify that $g(n+3)=3g(n+2)-3g(n+1)+g(n)$.

If you do not want to use the induction form indicated previously to notice that:
$F(n+3)-3F(n+2)+3F(n+1)-F(n) = (F(n+3)-2F(n+2)+F(n+1))-(F(n+2)-2F(n+1)+F(n))$

We can demonstrate that $(F(n+2)-2F(n+1)+F(n))$ is divisible by p^3 , we would need to prove that $(F(2)-2F(1)+F(0))$ is divisible by p^3 .

If we proved that $(F(n+2)-2F(n+1)+F(n))$ is divisible by p^3 , we can do the following :
$(F(n+2)-2F(n+1)+F(n)) = (F(n+2)-F(n+1))-(F(n+1)-F(n))$

We can demonstrate that $(F(n+1)-F(n))$ is divisible by p^3 , we would need to demonstrate that $(F(1)-F(0))$ is divisible by p^3 . If we proved that $(F(n+1)-F(n))$ is divisible by p^3 , it would reduce to us to demonstrate that $F(0)$ is divisible by p^3 , to prove that $F(n)$ is divisible by p^3 .

Summarizing, if we do the above and we prove that: $(F(2)-2F(1)+F(0))$, $(F(1)-F(0))$ and $F(0)$ are divisible by p^3 , we can prove that F(n) is divisible by p^3 , for all integer $n\geq 0$, but if we proved that $F(0)$,

$F(1)$ and $F(2)$ are divisible by p^3 , we demonstrated that $(F(2)-2F(1)+F(0))$,$(F(1)-F(0))$ and $F(0)$ are divisible by p^3 .

Therefore both approaches are enough similarities.

Other ways to solve Problem 1 can exist, perhaps easier , but is important to give more alternatives , since the fact to limit itself a certain method when we faced a problem, can lead to us simply to a mistaken demonstration or to not being able to solve it.

Hint Problem 2 :

Let $F(n)=1+2^{2n}+3^{2n}+2((-1)^{FIB(n)}+1)$ .Prove that $(F(n+3)-F(n))$ is divisible by 7. Divide the original problem in three problems: n of the form $3m$,n of the form $3m-1$ and n of the form $3m-2$. Soon apply induction on m for each one of the problems.

A sophisticated form to solve the problem is demonstrating that $(F(n)+F(n+1)+F(n+2))$ is divisible by 7(using the principle of weak induction) and later to use the following form of induction:

1. The property is true for $n=1$ and $n=2$.
2. If the property is true for n and $(n+1)$ implies that the property is true for $(n+2)$.

Hint Problem 3 :

See hint for problem 2.Let $F(n)=2(2^{2n}+5^{2n}+6^{2n})+3(-1)^{n+1}((-1)^{FIB(n)}+1)$

For the second asked for demonstration prove that: $(F(n)-F(n+1)+F(n+2))$ is divisible by 13 or that $(F(n)^2+F(n+1)^2+F(n+2)^2)$ is divisible by 13.

Notice that if $F(n)^2$ is divisible by 13, $F(n)$ as well.

Solution Problem 4 :

We will use the following induction form:

1. The property is true for $n=0$ and $n=1$.
2. If the property is true for n and $(n+1)$, then is true for $(n+2)$.

Follow straight from statement that the property is true for $n=0$.(Later prove that the property is true for $n=1$.)

Of the generalized Fermat's little theorem is deduced that:

$$(a_i^{(b-1)(c-1)}-1)^2=a_i^{2(b-1)(c-1)}-2a_i^{(b-1)(c-1)}+1\equiv 0 \pmod{b^2c^2}$$
$$(a_i^{2(b-1)(c-1)}-2a_i^{(b-1)(c-1)}+1)a_i^{1+n(b-1)(c-1)}\equiv 0 \pmod{b^2c^2}$$
$$(a_i^{1+(n+2)(b-1)(c-1)}-2a_i^{1+(n+1)(b-1)(c-1)}+a_i^{1+n(b-1)(c-1)})\equiv 0 \pmod{b^2c^2}$$
$$(\sum_{i=1}^{m}(a_i^{1+(n+2)(b-1)(c-1)}-2\sum_{i=1}^{m}a_i^{1+(n+1)(b-1)(c-1)}+\sum_{i=1}^{m}a_i^{1+n(b-1)(c-1)})\equiv 0 \pmod{b^2c^2}$$

Therefore:

$$(F(n+2)-2F(n+1)+F(n))\equiv 0 \pmod{b^2c^2} \tag{1}$$

We will prove that:

$$F(n)\equiv nF(1) \pmod{b^2c^2} \tag{2}$$

We will use the induction form indicated previously. For $n=0$ we must demonstrate that $F(0)\equiv 0F(1) \pmod{b^2c^2}$,which is certain according to the statement. For $n=1$ we must demonstrate that $F(1)\equiv 1F(1) \pmod{b^2c^2}$, which is equivalent to demonstrate that$0\equiv 0 \pmod{b^2c^2}$, which is fulfilled clearly for positive whole numbers b and c.

Let us suppose that the property is true for n and $(n+1)$.We will prove that the property also is true for $(n+2)$.

In effect, of the relation (1) and applying the inductive hypothesis is deduced that:

$$(F(n+2)-2(n+1)F(1)+nF(1))\equiv 0 \pmod{b^2c^2}$$

$$F(n+2) \equiv (n+2)F(1) \pmod{b^2c^2}$$

Now we will demonstrate that there exists n so that $F(n)$ is divisible by $(bc)^2$ and n is not divisible by (bc)(If we demonstrated the above we can prove easily that $F(1)$ is divisible by $(bc)^2$).

We will prove that there exists n distinct from 0 so that $(1+(b-1)(c-1)n)$ is divisible by bc and that n is not divisible by bc.

If we prove the first can to prove that, using the second assumption given in the statement, for that n , $F(n)$ is divisible by $(bc)^2$;and if prove that n is not divisible by bc, can to use the result (2) for prove that $F(1)$ is divisible by $(bc)^2$.

We will prove that there exists n_b in such a way that $(1+(b-1)(c-1)n_b)$ is divisible by b and that there exists n_c in such a way that $(1+(b-1)(c-1)n_c)$ is divisible by c.

In effect , to multiply $(b-1)(c-1)$ by $n =$ 1,2 ,...,$(b-1)$ the remainders of the division by b are all different .

We guess that for $n = r$ and $n = s$ the remainders are equal with r and s positive integers $\leq (p-1)$ and r distinct from s. Therefore $(r-s)(b-1)(c-1)$ is divisible by b, but $(r-s)$, $(b-1)$ and $(c-1)$ are not divisible by b (r and s are both $< b$, hence their difference is $< b$ and either 0 ,because r is different from s; $(b-1)$ is not divisible by b, since 1 is not divisible by b ; and $(c-1)$ is not divisible by b under the statement).

In conclusion there exists a contradiction and therefore if the remainders are equal ,then $r = s$ and there exists a $n < b$ so that the remainder is equal to $(b-1)$. Therefore for that n,$(1+(b-1)(c-1)n)$ is divisible by b.

For c is a similar demonstration.

Now we need to find a n so that $(1+(b-1)(c-1)n)$ is divisible by bc.

$1+(b-1)(c-1)(n_b+k_bb) = 1+(b-1)(c-1)(n_c+k_cc)$

$(n_b+k_bb) = (n_c+k_cc)$

$k_bb = (n_c-n_b)+k_cc$

If $n_c = n_b$ obviously $n = n_c = n_b$.

If n_c is distinct from n_b , there exists k_c so that if to the remainder of the division of k_cc by b we added the remainder of the division of (n_c-n_b) by b the result is b. The demonstration is similar to the done before(if we divide k_cc by b for $k_c = 1, 2, \ldots, b-1$ obtain $b-1$ different remainders that, evidently, are $1, 2, \ldots, b-1$.)

Therefore we proved that there exists n so that $(1+(b-1)(c-1)n)$ is divisible by bc,

The n found is not divisible by bc, since of be divisible by bc it would imply that 1 is divisible by ab which is a contradiction.

We proved that $F(0)$ is divisible by b^2c^2 and with the above result we proved that $F(1)$ is divisible by b^2c^2 .

Let us suppose that $F(n)$ and $F(n+1)$ are both divisible by b^2c^2, from the result (1) is deduced that $F(n+2)$ also is divisible by b^2c^2.

Therefore: $F(n)$ is divisible by b^2c^2 , for all non-negative integers n .

Solution Problem 5 :

Part a

We will prove, previously, that $(a^2b^2+a^2c^2+b^2c^2)$ is divisible by p^2. It is known that $(a^2+b^2+c^2)$ is divisible by p, implies that $(a^2+b^2+c^2)^2$ is divisible by p^2.

But $(a^2+b^2+c^2)^2$ is equal to $(a^4+b^4+c^4+2(a^2b^2+a^2c^2+b^2c^2))$.

Hence :

$(a^4+b^4+c^4+2(a^2b^2+a^2c^2+b^2c^2))$ is divisible by p^2.(1)

On the other hand :

$$\begin{aligned} a+b &= c \\ (a+b)^2 &= c^2 \\ a^2+2ab+b^2 &= c^2 \\ a^2+b^2-c^2 &= -2ab \end{aligned}$$

$$\begin{aligned}(a^4+b^4+c^4+2(a^2b^2-a^2c^2-b^2c^2)) &= 4a^2b^2\\(a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2)) &= 0 \qquad (2)\end{aligned}$$

Reducing (2) less (1) we have that $4(a^2b^2+a^2c^2+b^2c^2)$ is divisible by p^2, but as p is odd number is deduced that $(a^2b^2+a^2c^2+b^2c^2)$ is divisible by p^2.

Now we return to the original problem.

For $n=1$ we have that: $a^{6-4}+b^{6-4}+c^{6-4}=a^2+b^2+c^2$

Which is divisible by p according to statement.

Let us suppose that the statement is true for $(n-1)$. We will demonstrate that the property is true for n.

Notice that: $(a^2+b^2+c^2)(a^{6n-6}+b^{6n-6}+c^{6n-6})=$
$=a^{6n-4}+a^2b^{6n-6}+a^2c^{6n-6}+b^2a^{6n-6}+b^{6n-4}+b^2c^{6n-6}+c^2a^{6n-6}+c^2b^{6n-6}+c^{6n-4}$
$=a^{6n-4}+b^{6n-4}+c^{6n-4}+a^{6n-8}(a^2b^2+a^2c^2)+b^{6n-8}(b^2a^2+b^2c^2)+c^{6n-8}(c^2b^2+c^2a^2)$
$=a^{6n-4}+b^{6n-4}+c^{6n-4}+a^{6n-8}(a^2b^2+a^2c^2+b^2c^2-b^2c^2)+b^{6n-8}(b^2a^2+b^2c^2+a^2c^2-a^2c^2)+$
$c^{6n-8}(c^2b^2+c^2a^2+b^2a^2-b^2a^2)$
$=(a^{6n-4}+b^{6n-4}+c^{6n-4})+(a^{6n-8}+b^{6n-8}+c^{6n-8})(a^2b^2+a^2c^2+b^2c^2)+a^2b^2c^2(a^{6n-10}+b^{6n-10}+c^{6n-10})$
$=(a^{6n-4}+b^{6n-4}+c^{6n-4})+(a^{6n-8}+b^{6n-8}+c^{6n-8})(a^2b^2+a^2c^2+b^2c^2)+a^2b^2c^2(a^{6(n-1)-4}+b^{6(n-1)-4}+c^{6(n-1)-4})$

We know that $(a^2+b^2+c^2)$and $(a^2b^2+a^2c^2+b^2c^2)$ are divisible by p .

Hence if $(a^{6(n-1)-4}+b^{6(n-1)-4}+c^{6(n-1)-4})$ is divisible by p, $(a^{6n-4}+b^{6n-4}+c^{6n-4})$ as well.

Part b

For $n=1$ we have that:
$a^{6-2}+b^{6-2}+c^{6-2}=a^4+b^4+c^4$

Adding (1) more (2) (see part 1) we have that $2(a^4+b^4+c^4)$ is divisible by p^2 , which implies that $(a^4+b^4+c^4)$ is divisible by p^2 , since p is odd. Therefore the property is true for $n=1$.

Let us suppose that the statement is true for $(n-1)$.We will prove that the property is true for n.

Notice that:
$(a^2+b^2+c^2)(a^{6n-4}+b^{6n-4}+c^{6n-4})=$
$=a^{6n-2}+a^2b^{6n-4}+a^2c^{6n-4}+b^2a^{6n-4}+b^{6n-2}+b^2c^{6n-4}+c^2a^{6n-4}+c^2b^{6n-4}+c^{6n-2}$
$=a^{6n-2}+b^{6n-2}+c^{6n-2}+a^{6n-6}(a^2b^2+a^2c^2)+b^{6n-6}(b^2a^2+b^2c^2)+c^{6n-6}(c^2b^2+c^2a^2)$
$=a^{6n-2}+b^{6n-2}+c^{6n-2}+a^{6n-6}(a^2b^2+a^2c^2+b^2c^2-b^2c^2)+b^{6n-6}(b^2a^2+b^2c^2+a^2c^2-a^2c^2)+$
$c^{6n-6}(c^2b^2+c^2a^2+b^2a^2-b^2a^2)$
$=(a^{6n-2}+b^{6n-2}+c^{6n-2})+(a^{6n-6}+b^{6n-6}+c^{6n-6})(a^2b^2+a^2c^2+b^2c^2)+a^2b^2c^2(a^{6n-8}+b^{6n-8}+c^{6n-8})$
$=(a^{6n-2}+b^{6n-2}+c^{6n-2})+(a^{6n-6}+b^{6n-6}+c^{6n-6})(a^2b^2+a^2c^2+b^2c^2)+a^2b^2c^2(a^{6(n-1)-2}+b^{6(n-1)-2}+c^{6(n-1)-2})$

We know that $(a^2+b^2+c^2)$and $(a^{6n-4}+b^{6n-4}+c^{6n-4})$ are divisible by p .

Therefore $(a^2+b^2+c^2)(a^{6n-4}+b^{6n-4}+c^{6n-4})$ is divisible by p^2 . On the other hand $a^2b^2+a^2c^2+b^2c^2$ is divisible by p^2 .

Hence if $(a^{6(n-1)-2}+b^{6(n-1)-2}+c^{6(n-1)-2})$ is divisible by p^2 ,$(a^{6n-2}+b^{6n-2}+c^{6n-2})$ as well.

Part c

For $n=1$ we must prove that:

$a^2+b^2+c^2$ is divisible by p, which is true according to the statement.

$$(a^{2^n}+b^{2^n}+c^{2^n})^2=a^{2^{n+1}}+b^{2^{n+1}}+c^{2^{n+1}}+2((ab)^{2^n}+(ac)^{2^n}+(bc)^{2^n})$$

$$\begin{aligned}a+b &= c\\a^2+2ab+b^2 &= c^2 \qquad (1)\end{aligned}$$

On the other hand $a^2+b^2+c^2$ is divisible by p (2).

Subtract (2) less (1) we have that $2ab\equiv 2c^2 \pmod p$.

Thus, $ab\equiv c^2 \pmod p$, since p is odd.

$$\begin{aligned}a-c &= -b\\a^2-2ac+c^2 &= b^2 \qquad (1)\end{aligned}$$

On the other hand $a^2+b^2+c^2$ is divisible by p. (2)
Subtract (2) less (1) we have that $2ac \equiv -2b^2 \pmod{p}$.
Therefore, $ac \equiv -b^2 \pmod{p}$, since p is odd.

$$\begin{aligned} b-c &= -a \\ b^2+2bc+c^2 &= a^2 \qquad (1) \end{aligned}$$

On the other hand $a^2+b^2+c^2$ is divisible by p.(2)
Reducing (2) less (1) we have that $2bc \equiv -2a^2 \pmod{p}$.
Hence, $bce \equiv -a^2 \pmod{p}$, because p is odd.
Using the above results we have:
$(a^{2^n}+b^{2^n}+c^{2^n})^2 \equiv a^{2^{n+1}}+b^{2^{n+1}}+c^{2^{n+1}}+2((a^2)^{2^n}+(b^2)^{2^n}+(c^2)^{2^n}) \pmod{p}$
$(a^{2^n}+b^{2^n}+c^{2^n})^2 \equiv a^{2^{n+1}}+b^{2^{n+1}}+c^{2^{n+1}}+2(a^{2^{n+1}}+b^{2^{n+1}}+c^{2^{n+1}}) \pmod{p}$
$(a^{2^n}+b^{2^n}+c^{2^n})^2 \equiv 3(a^{2^{n+1}}+b^{2^{n+1}}+c^{2^{n+1}}) \pmod{p}$

Therefore if the property is true for n ,then for $n+1$ is true as well, since p is not divisible by 3.

Part d

For $n=1$ the property is true (see part 2).
Use the following results to complete the demonstration:

1. $(a^{4^n})^2+(b^{4^n})^2+(c^{4^n})^2$ is divisible by p (corollary of the property demonstrated in part 3.)
2. $a^{4^n}+b^{4^n}+c^{4^n}$ is divisible by p^2 (Inductive Hypothesis).

Make something equivalent to the made when we proved that $a^4+b^4+c^4$ is divisible by p^2.

Solution Problem 6 :

Part a

Proposed.

Part b

For $n=2$ we must prove that:
$F(2)^2+F(2+1)^2 = F(2\cdot 2+4)-F(2\cdot 2-3)$
$F(2)^2+F(3)^2 = F(8)-F(1)$
That is equivalent to prove that:
$6^2+7^2 = 86-1$, i.e. $36+49=85$, which is clearly certain.
Let us notice that:
$\sum_{i=1}^{n} F(i)^2 = F(n)F(n+1)-5 \qquad (1)$
$\sum_{i=1}^{n+1} F(i)^2 = F(n+1)F(n+2)-5 \qquad (2)$
Adding (1) and (2) we have:
$F(1)^2+\sum_{i=1}^{n}(F(i)^2+F(i+1)^2) = F(n+1)(F(n)+F(n+2))-10$
$=(F(n+2)-F(n))(F(n)+F(n+2))-10$
$=F(n+2)^2-F(n)^2-10$
On the other hand:
$F(n+1)^2+F(n+2)^2 = (F(n)^2+F(n+1)^2)+(F(n+2)^2-F(n)^2)$
$=\sum_{i=1}^{n}(F(i)^2+F(i+1)^2)+((F(n)^2+F(n+1)^2)+1^2+10$
$=F(1)^2+F(2)^2+\sum_{i=2}^{n}(F(i)^2+F(i+1)^2)+2((F(n)^2+F(n+1)^2)+1^2+10$
Using the inductive hypothesis corresponding at the of strong induction principle and replacing the values of $F(1)$ and $F(2)$ we have:
$F(n+1)^2+F(n+2)^2 = 1+36+\sum_{i=2}^{n}(F(2i+4)-F(2i-3))+(F(2n+4)-F(2n-3))+11$
$=\sum_{i=2}^{n}((F(2i+5)-F(2i+3))-(F(1)+\sum_{i=3}^{n}(F(2i-2)-F(2i-4))+(F(2n+4)-F(2n-3))+48$
$=(F(2n+1)-F(7))-(F(2n)-F(2))+(F(2n+4)+F(2n-3))+48-F(1)$
$=(F(2n+5)-53)-(F(2n-2)-6)+F(2n+4)-F(2n-3)+48-1$
$=F(2n+5)-(2n-2)+F(2n+4)-F(2n-3)$
$=(F(2n+4)+F(2n+5))-(F(2n-3)+F(2n-2))$

$= F(2n+6) - F(2n-1)$
$= F(2(n+1)+4) - F(2(n+1)-3)$

Therefore the property is true for all integer n greater than 1.

Solution Problem 7 : Let:

$$F(n) = \sum_{k=1}^{p} k^{2^n}$$

For $n = 1$ we have: $F(1) = \frac{p(p+1)(2p+1)}{6}$, that, evidently, is divisible by $(2p+1)$, since 6 cannot divide $2p+1$(prime greater than 3).

Let us suppose that the property is true n.We will prove that is true for $n+1$ as well.

Proof 1

Let r_k be the remainder of the division of k^2 divided by (2p+1).Let R_k be equal to r_k ,if r_k is smaller or equal to p; and R_k equal to $(2p+1-r_k)$ if r_k is greater than p.

Therefore $k^2 \equiv \pm R_k \pmod{(2p+1)}$. Using binomial theorem we have:

$$\sum_{k=1}^{p} k^{2^{n+1}} = \sum_{k=1}^{p} k^{2(2)^n} \equiv \sum_{k=1}^{p} {R_k}^{2^n} \pmod{(2p+1)}$$

Now we need to prove that the R_k are all different .

Let a,b be positive integers smaller than or equal to p.

$a^2 \equiv \pm R_a \pmod{(2p+1)}$
$b^2 \equiv \pm R_b \pmod{(2p+1)}$

We guess that $R_a = R_b = R$ with a distinct from b:

Case 1: Let us suppose same sign.Suppose positive sign.In the case of negative sign the demonstration is similar. $a^2 \equiv R(1) \pmod{(2p+1)}$
$b^2 \equiv R(2) \pmod{(2p+1)}$

Subtracting (1) less (2) we have:

$a^2 - b^2 \equiv 0 \pmod{(2p+1)}$
$(a-b)(a+b) \equiv 0 \pmod{(2p+1)}$

The above result implies that $2p+1$ divides $(a-b)$ or $(a+b)$, since $2p+1$ is prime.But as $(a-b)$ as $(a+b)$ are not divisible by $2p+1$, a and b are smaller or equal to p and therefore the absolute value of their difference is minor than $2p+1$ and cannot either be zero, since a is different from b.

On the other hand $(a+b)$ also is smaller than $2p+1$, because the greater value of $(a+b)$ it is obtained when one of the values is p and the other $p-1$, i.e. when their sum is $2p-1$.

Thus, if there exist a and b, they cannot have the same sign.

Case 2:different sign.We guess that the negative sign corresponds to b. The opposite case is similar.

$a^2 \equiv R \pmod{(2p+1)}$ (3)
$b^2 \equiv -R \pmod{(2p+1)}$ (4)

Let us elevate (3) and (4) to p.We remember that p is odd.

$a^{2p} \equiv R^p \pmod{(2p+1)}$
$b^{2p} \equiv -R^p \pmod{(2p+1)}$
$a^{2p} - 1 \equiv R^p - 1 \pmod{(2p+1)}$
$b^{2p} - 1 \equiv -R^p - 1 \pmod{(2p+1)}$

The previous result implies that $R^p - 1$ and $-R^p - 1$ are divisible by $2p+1$ (using Fermat's little theorem) and therefore their sum as well, but their sum is -2, i.e. if we suppose that $R_a = R_b = R$ with a distinct from b this implies that -2 is divisible by $2p+1$, which is a contradiction. Therefore if a different from b it implies that R_a is different from R_b .

We finished proving that the R_k are all different and moreover from the definition of R_k is deduced that:

$$\sum_{k=1}^{p} k^{2^n} = \sum_{k=1}^{p} {R_k}^{2^n}$$

Therefore $F(n+1) \equiv F(n) \pmod{(2p+1)}$.Hence if $F(n)$ is divisible by $2p+1$, $F(n+1)$ as well.

Proof 2

$$(F(n))^2 = F(n+1) + 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}(ij)^{2^n} \tag{3}$$

Let r_{ij} be the remainder of the division of (ij) divided by $(2p+1)$.

Let R_{ij} be equal to r_{ij} ,if r_{ij} is smaller than or equal to p; and R_{ij} equal to $(2p+1-r_k)$ if r_k is greater than p.

Therefore $(ij) \equiv \pm R_{ij} \pmod{(2p+1)}$.

$$(F(n))^2 \equiv F(n+1) + 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}(R_{ij})^{2^n} \pmod{(2p+1)}$$

$$(F(n))^2 \equiv F(n+1) + 2\sum_{i=1}^{p} c_k k^{2^n} \pmod{(2p+1)}$$

Where:

$$\sum_{k=1}^{p} c_k = \frac{p(p-1)}{2} \tag{4}$$

Several forms exist to prove the statement before, but I believe that the easiest is the following : The left side of the equality (3) has p^2 terms corresponding to $(F(n))^2$. The right side has p terms corresponding to $F(n)$ more twice the number of terms that we wished to find.Hence finding the unknown quantity we obtain the looked for result.

We are going to demonstrate that to $c_k \leq \frac{(p-1)}{2}$ which is equivalent to prove that given first p natural numbers we can form at the most $\frac{(p-1)}{2}$ pairs of numbers d and e such that $d \cdot e \equiv \pm k \pmod{(2p+1)}$.

We know that p is odd therefore $(p-1)$ is even .Hence at most we can form$\frac{(p-1)}{2}$ pairs. In order to be able to form more pairs we must occupy the number that exceeded to us and return to occupy another number already used.But another combination with two numbers or do the above would imply the following:

$$d \cdot e \equiv \pm k \pmod{(2p+1)} \tag{5}$$

$$d \cdot f \equiv \pm k \pmod{(2p+1)} \tag{6}$$

with f different from e

We have two cases: same sign or different sign.

If they have same sign we can subtract (6)less (5). $d(f-e) \equiv 0 \pmod{(2p+1)}$

If they have different sign we can add (6) more(5). $d(f+e) \equiv 0 \pmod{(2p+1)}$

The above results imply that $2p+1$ divides d,$(f-e)$ or $(f+e)$, since $2p+1$ is prime.

But d is not divisible by $2p+1$, since it is a positive integer smaller than $2p+1$ and as $(f-e)$ as $(f+e)$ no can be divisible by $2p+1$, f and e are smaller than or equal to p and therefore the absolute value of their difference is minor than $2p+1$ and cannot either be zero, since f is distinct from e.

On the other hand $(f+e)$ also it is smaller that $2p+1$, because in the greater value of $(f+e)$ is obtained when one of the values is p and the other $p-1$, i.e. when their sum is $2p-1$.

To guess that we can form more of $\frac{(p-1)}{2}$ pairs takes us to a contradiction.We can form at most $\frac{(p-1)}{2}$ pairs.

Therefore $c_k \leq \frac{(p-1)}{2}$ for $k = 1,\ldots,p$. But c_k satisfy the equality (4).Hence the only possible value for $c_k k = 1,\ldots,p$ is $\frac{(p-1)}{2}$, since otherwise the equality would not be satisfied.

Of the previous result we deduce that:

$$(F(n))^2 \equiv (F(n+1) + (p-1)F(n)) \pmod{(2p+1)}$$

$$F(n+1) \equiv -F(n)(F(n)-p+1) \pmod{(2p+1)}$$

Therefore if $F(n)$ is divisible by $2p+1$, $F(n+1)$ as well.

Proof 3

Let a be a positive integer smaller than or equal to p,and greater than 1 , prove that: $(a^{2^n}-1)\sum_{k=1}^{p} k^{2^n}$ is divisible by $2p+1$.Then prove that $(a^{2^n}-1)$ it is possible to be expressed like the product of sum of squares by $(a-1)(a+1)$.

Later demonstrate that $2p+1$ cannot divide the sum of two squares (See Proof 1) and deduces it asked for.

Hints in order to prove a more general property:

Now we will give hints to prove that $\sum_{k=1}^{p} k^{2n}$ is divisible by $2p+1$, except for n multiple of p. Notice that the property just proved is a particular case of this more general property.

We will concentrate in first p cases, since:

$$\sum_{k=1}^{p} k^{2(n+p)} \equiv \sum_{k=1}^{p} k^{2n} \pmod{(2p+1)}$$

The above expression is a direct result from the Fermat's little theorem.

On the other hand, using the same previous theorem, it is easy to prove that for n multiple of p: $\sum_{k=1}^{p} k^{2n}$ is of the form $(2p+1)m+p$ and therefore is not divisible by $2p+1$.

Prove that:

$$(\sum_{k=1}^{p} k^{2n})((\sum_{k=1}^{p} k^{2n})-p) \equiv 0 \pmod{(2p+1)} \tag{7}$$

Soon prove that:

$$(\sum_{k=1}^{p} k^{2} + \sum_{k=1}^{p} k^{4} + \ldots + \sum_{k=1}^{p} k^{2(p-1)}) \equiv 0 \pmod{(2p+1)} \tag{8}$$

Use the following trick :Rearrange the terms forming geometric progressions.

Of (7) it is deduced that$\sum_{k=1}^{p} k^{2n}$ is divisible by $2p+1$ or $\sum_{k=1}^{p} k^{2n}-p$ is divisible by $2p+1$.

Therefore: $\sum_{k=1}^{p} k^{2n} \equiv c_n p \pmod{(2p+1)}$ Where c_n is 0 or 1.

Replacing in the result (8) we have: $(c_1+c_2+\ldots+c_{p-1})p$ is divisible by $2p+1$.

Hence: $(c_1+c_2+\ldots+c_{p-1})$ is divisible by $2p+1$, since $2p+1$ is prime.

The above quantity is like minimum 0 and at most $(p-1)$, and thus the only possible value is zero, since otherwise $(c_1+c_2+\ldots+c_{p-1})$ no could be divisible by $2p+1$.

Therefore $\sum_{k=1}^{p} k^{2n}$ is divisible by $2p+1$ for $n = 1,2,\ldots,p-1$.

Hint Problem 8 :

For $n=1$ we must to prove that $4p+1$ does divide $\sum_{k=1}^{p} a_k^2$

Proof 1

Let b be a positive integer so that b is smaller than or equal to $2p$ and has the property: $b^{2p}+1$ is divisible by $(4p+1)$.

$$b^2\sum_{k=1}^{p} a_k^2 = \sum_{k=1}^{p}(b \cdot a_k)^2$$

Let B_k be the remainder of the division of $(b \cdot a_k)$ divided by (4p+1).

Let b_k be equal to B_k ,if r_k is smaller or equal to $2p$; and b_k equal to $((4p+1)-B_k)$ if B_k is greater than $2p$. Therefore $(b \cdot a_k) \equiv \pm b_k \pmod{(4p+1)}$.

Next to prove that the b_k are all different, belongs to the set of the first $2p$ positive integers and they have the property: $b_k^{2p}+1$ is divisible by $(4p+1)$.

So :

$$b^2\sum_{k=1}^{p} a_k^2 \equiv \sum_{k=1}^{p} b_k^2 \pmod{(4p+1)}$$

and hence:

$$(b^2\sum_{k=1}^{p}a_k^2+\sum_{k=1}^{p}a_k^2)\equiv(\sum_{k=1}^{p}b_k^2+\sum_{k=1}^{p}a_k^2)\pmod{(4p+1)}$$

Notice that $(\sum_{k=1}^{p}b_k^2+\sum_{k=1}^{p}a_k^2)$ is the summation of the first $2p$ positive integers, i. e. $\frac{(2p(2p+1)(4p+1))}{6}$, that, evidently, is divisible by $(4p+1)$, since 6 cannot divide $4p+1$(prime greater than 3).

Therefore:

$$(b^2+1)\sum_{k=1}^{p}a_k^2\equiv 0\pmod{(4p+1)}$$

You can select b_r and b_s (b_r distinct from b_s) so that at the most one of them has the property (b^2+1) is divisible by $(4p+1)$.

Prove it and complete the proof.

Proof 2

Make something similar to proof 1,but with an a so that $a^{2p}-1$ is divisible by $(4p+1)$(a distinct from 1).

For the rest of the solution see Solution Problem 7.

Hint Problem 9 :

Consider the quadratic residues of 13.Moreover it is easy to prove that :
$4^{2n-1}+9^{2n-1}$ is divisible by 13.

Hint Problem 10 :

Prove that $8^{2^n}-3^{2^n}$ is divisible by 13 and is not divisible by 13^2.

Hint Problem 11 :

See hint Problem 1.

Hint Problem 12 :

Guess and prove that for every positive integer n :

$$f(a+n\cdot b)\equiv k^n f(a)\pmod{p}.$$

Next to use Euler's theorem.

Hint Problem 13 :

Let: $f(n)=1+2^{4n+2}+3^{4n+2}+4^{4n+2}+5^{4n+2}+6^{4n+2}$

Notice that $f(n)=(1+5^{2(2n+1)})+(2^{2(2n+1)}+3^{2(2n+1)})+(4^{2(2n+1)}+6^{2(2n+1)})$
$=(1+5^{2(2n+1)})+(2^{2(2n+1)}+3^{2(2n+1)})+2^2(2^{2(2n+1)}+3^{2(2n+1)})$
$=(1+5^{2(2n+1)})+5(2^{2(2n+1)}+3^{2(2n+1)})$

Next demonstrate that $(1+5^2(2n+1))$ and $(2^{2(2n+1)}+3^{2(2n+1)})$ are divisible by 13.

Another solution is divide the original problem in three problems: n of the form $3m$, n of the form $3m-1$ and n of the form $3m-2$.

Hint Problem 14 :

Let $f(n)=(2(3^{4n+3}+4^{4n+3})-25n^2+65n+68)$.

Consider $f(n+1)-34f(n)$.Another solution is to use Hint Problem 1.

Hint Problem 15 :

Let:
$F(n)=(2^{2^n}+3^{2^n}+5^{2^n})$

Notice that:
$F(n+2)=(a^{2^{n+2}}+b^{2^{n+2}}+c^{2^{n+2}})=((2^4)^{2^n}+(3^4)^{2^n}+(5^4)^{2^n})$
$=(16^{2^n}+81^{2^n}+625^{2^n})=((19-3)^{2^n}+(19\cdot 4+5)^{2^n}+(19\cdot 33-2)^{2^n})$

Using binomial theorem , we have: $F(n+2)\equiv F(n)\pmod{19}$

In order to complete the demonstration to divide the original problem in two problems: odd n and even n.Also we can demonstrate that $(F(n)+F(n+1))$ is divisible by 19 and later to deduce that $F(n)$ is divisible by 19. See indication Problem 2.

Solution Problem 16 :

For $n=1$ we must prove that:
$g(1)=(f(1)+f(2))(2a-1)\cdot a^0$

In effect, $g(1)=f(3)+af(2)+(a-1)f(1)=(a-1)f(2)+af(1)+af(2)+(a-1)f(1)$
$=(2a-1)(f(1)+f(2))=(2a-1)(f(1)+f(2))\cdot a^0$

Proof 1

$g(n+1)=f(n+3)+af(n+2)+(a-1)f(n+1)$
$a\cdot g(n)=af(n+2)+a^2f(n+1)+a(a-1)f(n)$

Hence:
$g(n+1)-a\cdot g(n)=$
$=f(n+3)+af(n+2)+(a-1)f(n+1)-af(n+2)-a^2f(n+1)-a(a-1)f(n)$
$=f(n+3)-(a^2-a+1)f(n+1)-a(a-1)f(n)$

But $f(n+3)=(a-1)f(n+2)+af(n+1)=(a-1)((a-1)f(n+1)+af(n))+af(n+1)$
$=((a-1)^2+a)f(n+1)+a(a-1)f(n)$
$=(a^2-a+1)f(n+1)+a(a-1)f(n)$

Therefore:
$f(n+3)-(a^2-a+1)f(n+1)-a(a-1)f(n)=0$, and therefore $g(n+1)-a\cdot g(n)=0$ that is equivalent to $g(n+1)=a\cdot g(n)$

Applying the inductive hypothesis we have:
$g(n+1)=a\cdot(f(1)+f(2))\cdot(2a-1)\cdot a^{(n-1)}$
$=(f(1)+f(2))\cdot(2a-1)\cdot a^{((n+1)-1)}$

Proof 2

$g(n)=f(n+2)+af(n+1)+(a-1)f(n)$
$=(a-1)f(n+1)+af(n)+af(n+1)+(a-1)f(n)=(2a-1)(f(n)+f(n+1))$

Therefore we can prove that $(f(n)+f(n+1))=(f(1)+f(2))a^{(n-1)}$

For $n=1$ is clear that $(f(1)+f(2))=(f(1)+f(2))a^{(1-1)}$

We know that $f(n+2)=(a-1)f(n+1)+af(n)$

Adding $f(n+1)$ to both sides of the equality we have: $f(n+2)+f(n+1)=a(f(n+1)+f(n))$

Applying the inductive hypothesis we have:
$f(n+2)+f(n+1)=a(f(1)+f(2))\cdot a^{(n-1)}$
$f((n+1)+1)+f(n+1)=(f(1)+f(2))\cdot a^{((n+1)-1)}$

Solution Problem 17 :

For n=1 we have: $f(3\cdot 1)+f(3\cdot 1+1)=f(3)+f(4)$
$f(3)=3(1+1)+1=7$
$f(4)=3(7+1)+1=25$

Hence:
$f(3)+f(4)=7+25=32$, that, evidently, is divisible by 32.
$f(3(n+1))+f(3(n+1)+1)=f(3n+3)+f(3n+4)$
$=f(3n+3)+3(f(3n+3)+f(3n+2))+1=4f(3n+3)+3f(3n+2)+1$
$=4(3(f(3n+2)+f(3n+1))+1)+3f(3n+2)+1$
$=15f(3n+2)+12f(3n+1)+5$
$=15(3(f(3n+1)+f(3n))+1)+12f(3n+1)+5$
$=12f(3n+1)+20+45(f(3n+1)+f(3n))$
$=4(3f(3n+1)+5)+45(f(3n+1)+f(3n))$

If we proved that $3f(3n+1)+5$ is divisible by 8, we can complete the proof.

For $n=1$, we must prove that:
$3f(3\cdot 1+1)+5$ is divisible by 8.
$3f(4)+5=3\cdot 25+5=80$, that, clearly, is divisible by 8.
$3f(3(n+1)+1)+5=3f(3n+4)+5=3(3(f(3n+3)+f(3n+2))+1)+5$
$=3(3(3(f(3n+2)+f(3n+1))+1+f(3n+2))+1)+5$
$=3(12f(3n+2)+9f(3n+1)+2)+5$
$=36f(3n+2)+27f(3n+1)+17$
$=36f(3n+2)+9(3f(3n+1)+5)-28$

$= 4(9f(3n+2)-7)+9(3f(3n+1)+5)$

We would need to prove that $(9f(3n+2)-7)$ is divisible by 2, or that $(f(3n+2)-1)$ is divisible by 2.

We will prove that (f(3n+2)-1) is divisible by 2.

For $n=1$, we must prove that:
$f(3\cdot 1+2)-1$ is divisible by 2. $f(5)-1=97-1=96$, that, evidently, is divisible by 2. $f(3(n+1)+2)-1 = f(3n+5) = 3(f(3n+4)+f(3n+3))+1-1$
$= 3(3(f(3n+3)+f(3n+2))+1+f(3n+3))$
$= 12f(3n+3)+9f(3n+2)+3$
$= 12(f(3n+3)+1)+9(f(3n+2)-1)$

Therefore if f(3n+2)-1 is divisible by 2, f(3(n+1)+2)-1 as well.

Therefore (9f(3n+2)-7) is divisible by 2, with which we can complete the demonstration of which $3f(3n+1)+5$ is divisible by 8 and therefore to finish demonstrating that for all positive integers n: $(f(3n)+f(3n+1))$ is divisible by 32.

Hint Problem 18 :

Prove that $f(n) \equiv n \cdot f(1) \pmod{p^2}$, using hint of problem 20 (but base cases are 0 and 1).Next to prove that $f(100)$ is divisible by p^2 and hence $f(1)$ is divisible by p^2.

Hint Problem 19 :

See Hint Problem 1.

Hint Problem 20 :

Use the following form of induction:

1. The property is true for $n=1$ and $n=2$.
2. If the property is true for n and $(n+1)$, then the property is true for $(n+2)$.

Hint Problem 21 :

Notice that we must determine $S_1(n)$,$S_2(n)$,$S_3(n)$,$S_4(n)$ and $S_5(n)$ such that:

$$F(S_1(n)) + F(S_1(n)) = F(S_3(n))(F(S_4(n)) + F(S_5(n)))$$

Consider a few values of n and take the successive differences for each $S_i(n)$. Use Binet's formula.

Hint Problem 22 : See Hint Problem 21.

Hint Problem 23 :

Let $f(n) = a^{(4+(p-1)n)} + b^{(4+(p-1)n)} + (a+b)^{(4+(p-1)n)}$.

Prove that $f(n) \equiv ((1-n)f(0)+nf(1)) \pmod{p^2})n \geq 0$.Use the following form of induction:

1. The property is true for $n=0$ and $n=1$.
2. If the property is true for n and $(n+1)$, then the property is true for $(n+2)$.

Let $g(k) = a^{(6k-2)} + b^{(6k-2)} + (a+b)^{(6k-2)}$.From problem 5(b), we know that $g(k)$ is divisible by p^2 for every positive integer k.In particular for $k=1$ and $k=p$,but notice that $g(1)=f(0)$ and $g(p)=f(6)$.Hence $f(0)$ and $f(6)$ are both divisible by p^2. So $f(6) \equiv 6f(1) \pmod{p^2}$,then $f(1)$ is divisible by p^2 (p is relatively prime to 6) and therefore $f(n) \equiv 0 \pmod{p^2}$ for each $n \geq 0$.

Hint Problem 24 :

See hint of Problem 7 and use the following result: a^2-ab+b^2 and a^2+ab+b^2 are divisible only by primes of the form $6k+1$ or by 3 (a relatively prime to b).

Hint Problem 25 :

Show that:$\sum_{i=1}^{n} F(i)^2 = F(n)F(n+1)$ and see solution Problem 6.Another solution is to use Binet's formula.

Hint Problem 26 :

Consider even n and otherwise and use the following:

$$\begin{aligned} F(n+10) &= 55F(n+1)+34F(n) \\ F(n+10) &= 11(5F(n+1)+3F(n))+F(n) \end{aligned}$$

Hint Problem 27 :
Let:
$F_i(n) = (n-0)\dots(n-(i-1))(n-(i+1))\dots(n-(k-1))$
Prove that:

$$F(n) \equiv \frac{F_0(n)F(0)}{F_0(0)} + \frac{F_1(n)F(1)}{F_1(1)} + \dots + \frac{F_{k-1}(n)F(k-1)}{F_{k-1}(k-1)} \pmod{p^k}$$

(Notice that for $k=2$,$F(n) \equiv ((1-n)F(0)+nF(1)) \pmod{p^2}$ and for $k=3$ $F(n) \equiv (\frac{(n-1)(n-2)}{2}F(0) - n(n-2)(1) + \frac{n(n-1)}{2}F(2)) \pmod{p^3})$
Use the form of induction that is indicated next:

1. The property is true for $n=0$, $n=1$,$n=2$,…,$n=k-1$
2. If the property is true for n, $(n+1)$,$(n+2)$,…,$(n+k-1)$,then the property is true for $(n+k)$.

It is easy to see that for$n=0$ $F(0) \equiv F(0) \pmod{p^k}$,for $n=1$ $F(1) \equiv F(1) \pmod{p^k}$,…, for $n=i$ $F(i) \equiv F(i) \pmod{p^k}$,...,and for $n=k-1$ $F(k-1) \equiv F(k-1) \pmod{p^k}$.Hence the property is true for $n=0$, $n=1$,$n=2$,…,$n=k-1$. For the inductive step notice that:

$$g(n) = \frac{F_0(n)F(0)}{F_0(0)} + \frac{F_1(n)F(1)}{F_1(1)} + \dots + \frac{F_{k-1}(n)F(k-1)}{F_{k-1}(k-1)}$$

Is a polynomial in the indeterminate n of degree at most (k-1).Therefore from calculus of finite differences we have that:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g(n+i) = 0$$

Complete this part of the proof using the fact that:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} F(n+i) \equiv 0 \pmod{p^k}$$

If $F(a_0)$,$F(a_1)$,...,$F(a_{k-1})$ are divisible by p^k , we have the following system:

$$\frac{F_0(a_0)F(0)}{F_0(0)} + \frac{F_1(a_0)F(1)}{F_1(1)} + \dots + \frac{F_{k-1}(a_0)F(k-1)}{F_{k-1}(k-1)} = c_0 p^k$$
$$\frac{F_0(a_1)F(0)}{F_0(0)} + \frac{F_1(a_1)F(1)}{F_1(1)} + \dots + \frac{F_{k-1}(a_1)F(k-1)}{F_{k-1}(k-1)} = c_1 p^k$$
$$\vdots$$
$$\frac{F_0(a_{k-1})F(0)}{F_0(0)} + \frac{F_1(a_{k-1})F(1)}{F_1(1)} + \dots + \frac{F_{k-1}(a_{k-1})F(k-1)}{F_{k-1}(k-1)} = c_{k-1} p^k$$

By Cramer's Rule $F(i) = \frac{\det(A_i)}{\det(A)}$, p^k is factor of $\det(A_i)$(also may be proved that $\frac{F_i(n)}{F_i(0)}$ is an integer for $i=0$,…,$(k-1)$ using the same form of induction indicated above).On the other hand it is easy prove that $\det(A)$ has the factors $(a_i - a_j)$ i different from j. The remaining factor of $\det(A)$ is a constant that can to be calculated evaluating $a_0 = 0$,…,$a_{k-1} = k-1$(the constant is unit fraction). We know that $F(i)$ is integer ,hence:

If $F(a_0)$,$F(a_1)$,...,$F(a_{k-1})$ are divisible by p^k where $(a_i - a_j)$ is not divisible by p for i different from j, then $F(i)$ is divisible by p^k for $i=0$,…,$(k-1)$.So $F(n) \equiv 0 \pmod{p^k}$.

Hint Problem 28 : Notice that $G_{n+2}(a) \equiv (G_{n+1}(a) + G_n(a)) \pmod{a^2-a-1}$(Polynomial division)
Prove that:$G_n(a) \equiv (F(n-1)G_0(a) + F(n)G_1(a)) \pmod{a^2-a-1}$
Use the following form of induction:

1. The property is true for $n = 0$ and $n = 1$.
2. If the property is true for n and $(n + 1)$, then the property is true for $(n + 2)$.

Then demonstrate that $G_0(a) = G_{11}(a) = 0$ (zero polynomial) ,hence $F(11)G_1(a)$ has the factor $a^2 - a - 1$,hence if r is a root of the polynomial $a^2 - a - 1$, we have the $89G_1(r) = 0$.So $G_1(r) = 0$(89 different from 0)and therefore: $G_n(a)$ is divisible by $a^2 - a - 1$ for every integer non-negative n. Also it is possible to determine,directly,that:

$$G_1(a) = -(a^2 - a - 1)(a^9 + a^8 + 2a^7 + 3a^6 + 5a^5 + 8a^4 + 13a^3 + 21a^2 + 34a + 55)$$

Hint Problem 29 : From Wilson's theorem there exits an integer a so that $a^2 \equiv -1 \pmod{4k+1}$.Prove that the integers from 1 to $2k$ can be arranged into k pairs such that the sum of the squares of each pair is divisible by $(4k + 1)$.The result follows from the well-known property: $x^{2n+1} + y^{2n+1}$ is divisible by $(x + y)$.

Hint Problem 30 : Prove that $G(n) \equiv (n(n-1)/2)G(2)) \pmod{p^3}$,then
$G(1001) \equiv 500500 \cdot G(2) \pmod{p^3}$ and conclude.